UNIT - 1

ELECTROSTATICS

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References:
1. N.O. Sadiku
2. K.D. Prasad.

UNIT – 1

Introduction to UNIT-1

→ The Sources (i/p) of electromagnetic fields are electric charges and the strength of a field at any point depends upon magnitude, position, velocity & acceleration of the charges involved.

→ An electrostatic field can be considered as a special case of electromagnetic field in which the sources are stationary. So that only the magnitude and position of the charges need to be considered.

→ Electrostatic field can be considered as electrostatics. There are two types of electrostatic charges
(i) Positive charge (i.e proton) and (ii) -Ve charge (i.e electron)

Fundamental relations of Electrostatic field :-

→ Between two charged particles (or) bodies there exists a force between them which is known as Electrostatic Force which tends to push them apart (or) pull them together depending on whether the charges on the particles are of same (or) opposite sign.

→ In electrostatic field, charges are stationary but the indivi-dual charges i.e electrons are not stationary. They have random velocities. From Macroscopic point of view stationary charge is meant the net charge.

→ Electrostatics divides materials in to two groups.

① Conductors ② Insulators

Conductors:- Bodies permitting charge (or) electricity to pass through them are called conductors ($\sigma >>> 1$)

Insulators:- Bodies which do not permit charge (or) electricity to pass through them are non-conductors (or) dielectrics (or) Insulators. ($\sigma << 1$)

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Introduction to UNIT-1

Vector Analysis can be classified in to two quantities.

1. Scalar quantity

→ A quantity that has only magnitude and algebraic sign is called Scalar quantity

→ Ex :- Physical quantities like mass, temp, time, density, electrostatic potential etc.

→ Scalar quantities are represented by italic letters

eg : A, B, c (or) a, b, c it indicates a magnitude only.

2. Vector quantity

→ A quantity that has both magnitude and direction is called Vector quantity.

→ Ex: velocity, force, electric field intensity, acceleration etc.

→ Vector quantities are represented by italic bold letters.

eg: **A**, **B**, **C** (or) **a**, **b**, **c** (or) $\vec{A}, \vec{B}, \vec{C}$

A → indicates magnitude

→ indicates the direction.

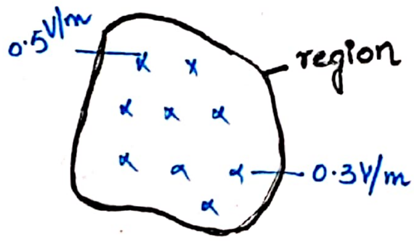
Field :- If at every point of region, there is a corresponding value of some physical function, the region is called a field.

The quantity is scalar the field is said to be a scalar field

" " " Vector " " " " " Vector field

Mathematically, A field is a function which describes the physical quantity at all points in space (or) in the region.

Field may be classified as



1. Scalar field

→ If the value of physical function at every point is scalar quantity then the field is scalar field

2. Vector field

→ If the value of physical function for each point is a vector quantity then the field is a vector field.

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→ In scalar field, physical quantity is completely specified by a single number

Ex: Mass, temp, density & time

→ In vector field physical quantity is completely specified by number & direction.

Ex: Velocity, gravitational force, electric field intensity.

Vector and Vector operations :- divided in to two categories

I. Vector Algebra

→ includes the addition, Subtraction, Multiplication of Vectors and the relations between the Components of Vectors in different Co-ordinate Systems.

II. Vector Analysis

→ includes the differential Vector operations, corresponding integral relations etc.

1. Vector Algebra :-

(i) Addition & Subtraction of Vectors : The sum of two (or) vectors is known as Vector Sum which is the resultant of the Vectors added. (The output is the sum of two vectors)

consider fig 1:

$\vec{A}$ & $\vec{B}$ are the two vectors.

→ vector $\vec{A}$ represents the displacement of movable point from (1) to (2)

→ Vector $\vec{B}$ represents the displacement from (2) to (3)

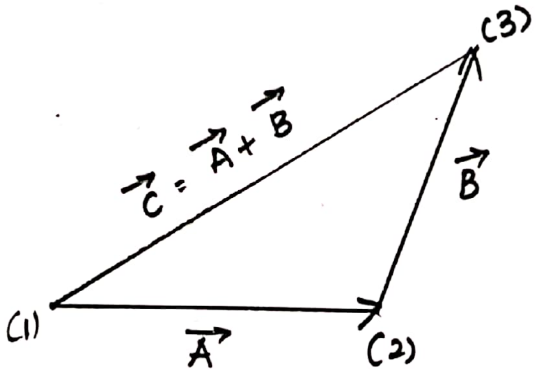


fig: Addition of '2' Vectors $\vec{A}$ & $\vec{B}$

The linear displacement from point (1) to Point (3) is the Sum of the two displacements (1, 2) & (2, 3) and is represented by another Vector ($\vec{C}$).

Vector Addition obeys commutation Law :

Consider two paths (1,4,3) & (1,2,3)

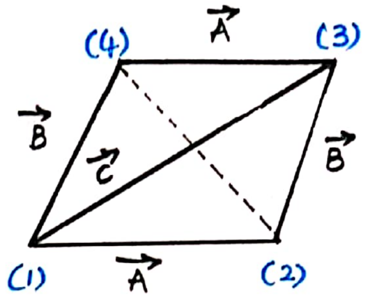


fig 2: Illustration of Commutation Law

1^{st} path → (1,4,3) the displacement from (1,3)

is $\vec{C} = \vec{B} + \vec{A}$ — ①

2^{nd} path → (1,2,3) the displacement from (1,3)

is $\vec{C} = \vec{A} + \vec{B}$ — ②

from ① & ② $\boxed{\vec{B} + \vec{A} = \vec{A} + \vec{B}}$

Commutation Law according to which sum of vectors is independent of the order of addition and the result remains the same even if the order of operation is changed i.e

$$\boxed{\vec{A} + \vec{B} = \vec{B} + \vec{A}}$$

Vector Addition is also called as parallelogram Law.

Vector Addition obeys Associative Law :

Consider the path (1,2,3) → the o/p is $\vec{A} + \vec{B}$

Consider the path (2,3,4) → the o/p is $\vec{B} + \vec{C}$

Path (1,3,4) → $(\vec{A} + \vec{B}) + \vec{C}$

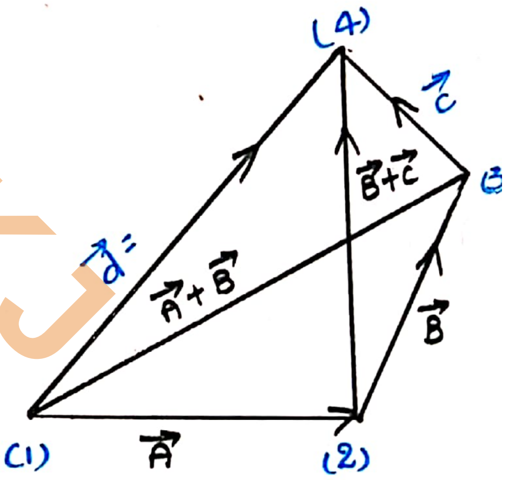


fig 3: Illustration of Associative Law

We are taking the output from (1 to 4)

Contains two paths

I Path → (1,3,4) } represented by
II Path → (1,2,4) }

another Vector

$$\vec{d} = (\vec{A} + \vec{B}) + \vec{C}$$

$$= (\vec{A} + (\vec{B} + \vec{C}))$$

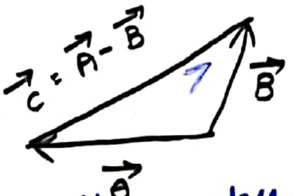
$$\therefore \boxed{(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})}$$

Vector Subtraction :

For ex: Vector $\vec{B}$ is subtracted from vector $\vec{A}$ by adding Vector $\vec{A}$ of the same magnitude of Vector $\vec{B}$ but in Opposite direction.

Negative of a Vector : is a vector of the same magnitude but with a reverse direction.

$$\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$$



Multiplication of Vectors : Can be done in three different ways.

1> A vector can be multiplied by a scalar and the resultant will be another vector.

i.e $\vec{A}b = \vec{C}$ → another vector

$\vec{A}$ → Vector, b → scalar

Graphically it can be represented by using two multiplication symbols.

(1) • dot

(2) × Cross

2> If the vector is multiplied by another vector in such a way that resultant is a scalar quantity then this is known as Scalar product (or) Dot product.

3> If the vector is multiplied by another Vector in such a way that the resultant is a vector quantity then this is known as Vector Product (or) cross product.

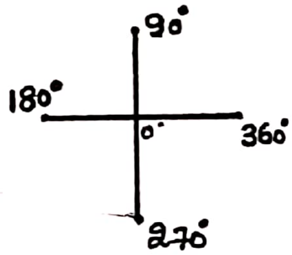
Scalar Product (or) Dot product :-

→ The Scalar (or) dot Product of two Vectors is defined as a Scalar quantity which is equal to the product of the magnitudes of the two given vectors multiplied by cosine of the angle between them.

The Scalar (OR) dot product of two Vectors $\vec{A}$ and $\vec{B}$ written as $\vec{A}\cdot\vec{B}$; is a scalar quantity having the magnitude equal to $AB\cos\theta$ Where 'θ' is the angle between vectors $\vec{A}$ and $\vec{B}$

i.e $\boxed{\vec{A}\cdot\vec{B} = |A||B|\cos\theta}$ $(0 \leq \theta \leq \pi)$

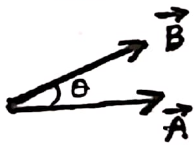
The Scalar Product of two Vectors $\vec{A}$ and $\vec{B}$ is represented by Putting dot (•) between two vectors and hence it is also called as dot product. Note that $\vec{A}\cdot\vec{B}$ is a scalar quantity not a Vector.



When $\boxed{\theta = 0}$ → $\cos\theta = 1$ → both the Vectors have the same direction.

$\boxed{\theta = 180°}$ → $\cos 180° = -1$ → two Vectors are in opposite direction

$\boxed{\theta = 90°}$ → $\cos 90° = 0$ → both are ⊥ar to each other



Dot Product obeys Commutative Law $\boxed{\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}}$

distributive Law

Consider fig:

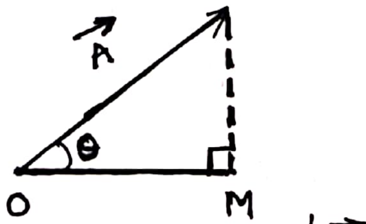
According to the dot product

$$\vec{A}\cdot\vec{C} = |A||C|\cos\theta$$

$$\cos\theta = \frac{OM}{\vec{A}}$$

$$\vec{A}\cdot\vec{C} = |\cancel{A}||C|\cdot\frac{OM}{\cancel{\vec{A}}}$$

$$= |C|\cdot OM$$



$\cos\theta = OM/\vec{A}$

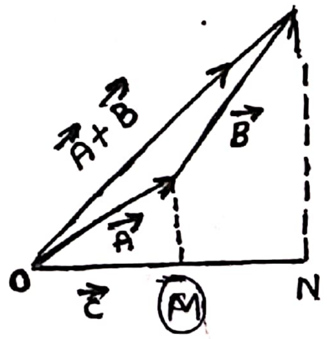
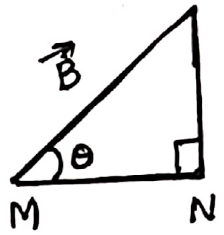


fig 7: illustration of distributive Law for scalar Multiplication.

$|||^{ly}$ consider $\vec{B}\cdot\vec{C} = |B||C|\cos\theta$

$\cos\theta = \dfrac{MN}{\vec{B}}$



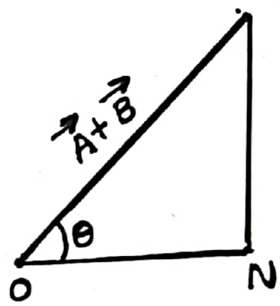
$$= \cancel{|B|}|C|\frac{MN}{\cancel{\vec{B}}} = |C|\,MN$$

$$\vec{A}\cdot\vec{C} + \vec{B}\cdot\vec{C} = |C|OM + |C|MN = ON|C| \quad \text{—— (1)}$$

Now consider

$$(\vec{A}+\vec{B})\cdot\vec{C} = |A+B||C|\cos\theta$$

from fig $\cos\theta = \dfrac{ON}{A+B}$



$\cos\theta = \dfrac{ON}{A+B}$

$$(\vec{A}+\vec{B})\cdot\vec{C} = \cancel{|A+B|}|C|\frac{ON}{\cancel{A+B}}$$

$$= ON|C| \quad \text{—— (2)}$$

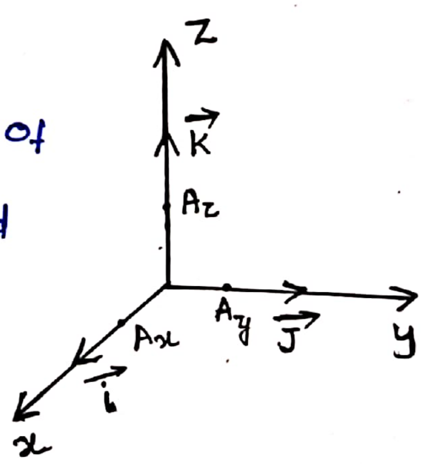
from (1) & (2)

$$\boxed{(\vec{A}+\vec{B})\cdot\vec{C} = \vec{A}\cdot\vec{C}+\vec{B}\cdot\vec{C}}$$ which Proves the distributive Law

Note: dot Product obeys Commutative & Associate & distributive Law

Consider the Vector $\vec{A}$ in three dimensions. x, y and z are the three co-ordinates. The three co-ordinates are mutually perpendicular to each other.

The vector $\vec{A}$ in co-ordinate system has three Components A_x, A_y & A_z.



→ A_x, A_y and A_z represent the magnitudes of the projections of the Vector $\vec{A}$ on x, y and z axes respectively.

→ Vector $\vec{A}$ in Co-ordinate System can be expressed as $\boxed{\vec{A} = \vec{i}A_x + \vec{j}A_y + \vec{k}A_z}$

Where $\vec{i}$, $\vec{j}$ and $\vec{k}$ are known as unit Vectors.

$\vec{i}$ is the unit vector along x-axis, $\vec{J}$ - Y axis & $\vec{K}$ - Z axis

UNIT VECTOR :- A Vector whose magnitude is unity in the Positive direction. It may also be represented as a_x, a_y & a_z.

→ Different notations are used to represent unit vectors.

$$\vec{i} \quad \vec{j} \quad \vec{k}$$

$$e_x \quad e_y \quad e_z$$

$$a_x \quad a_y \quad a_z$$

Relation between unit Vectors using dot product.

According to dot product of two vectors

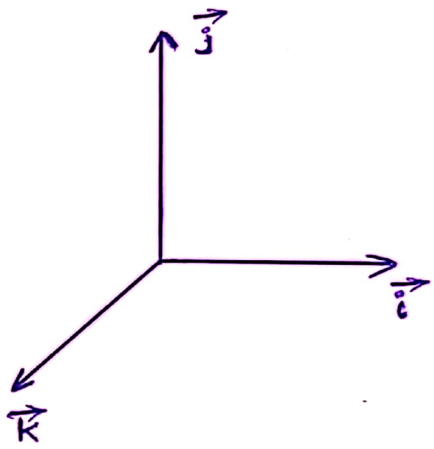
Case 1:- $\vec{i}\cdot\vec{i} = |i||i| \cos\theta$

from i to i the angle is '0°'

$\cos 0 = 1 \Rightarrow \vec{i}\cdot\vec{i} = |i||i|$

the magnitude of $|i| = 1$

$\therefore \vec{i}\cdot\vec{i} = \vec{j}\cdot\vec{j} = \vec{k}\cdot\vec{k} = 1$



Case 2:- $\vec{i}\cdot\vec{j} = |i||j| \cos\theta$

from i to j the angle is 90°

$\cos 90° = 0$

$\therefore \vec{i}\cdot\vec{j} = \vec{j}\cdot\vec{k} = \vec{k}\cdot\vec{i} = 0$

Note :- The dot product of same unit Vectors is equal to '1'

" " " different " " " " '0'

Dot Product of $\vec{A}\cdot\vec{B}$:

$\vec{A} = \vec{i}A_x + \vec{j}A_y + \vec{k}A_z$; $\vec{B} = \vec{i}B_x + \vec{j}B_y + \vec{k}B_z$

$$\vec{A}\cdot\vec{B} = (\vec{i}\cdot A_x + \vec{j}A_y + \vec{k}A_z)\cdot(\vec{i}B_x + \vec{j}B_y + \vec{k}B_z)$$

$$= \vec{i}\cdot\vec{i}\, A_x B_x + \vec{j}\cdot\vec{j}\, A_y B_y + \vec{k}\cdot\vec{k}\, A_z\cdot B_z$$

$$\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z$$

If the two Vectors are same then

$$\vec{A}\cdot\vec{A} = A_x A_x + A_y A_y + A_z A_z$$

$$|A||A|\cos\theta = A_x^2 + A_y^2 + A_z^2$$

$\theta = 0^\circ$

$$|A|^2 = A_x^2 + A_y^2 + A_z^2$$

$$\boxed{|A| = \sqrt{A_x^2 + A_y^2 + A_z^2}}$$

↓

magnitude of $\vec{A}$

Vector (or) cross product :-

The Vector (or) cross product of two Vectors is defined as a Vector whose direction is ⊥ar to the plane containing the two original Vectors and magnitude is equal to the product of magnitudes of those two vectors multiplied by the sine of the angle between them.

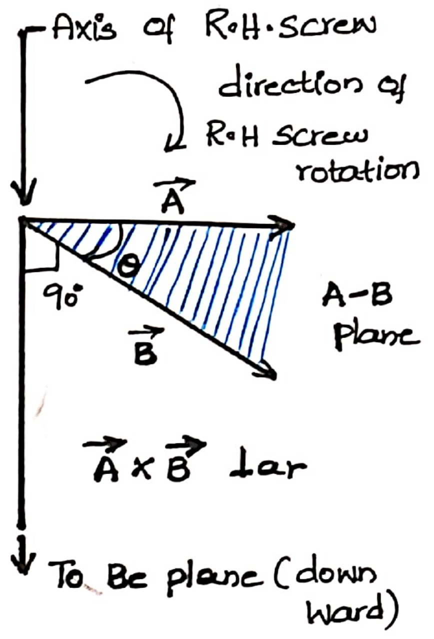
The cross product of two Vectors can be Written as

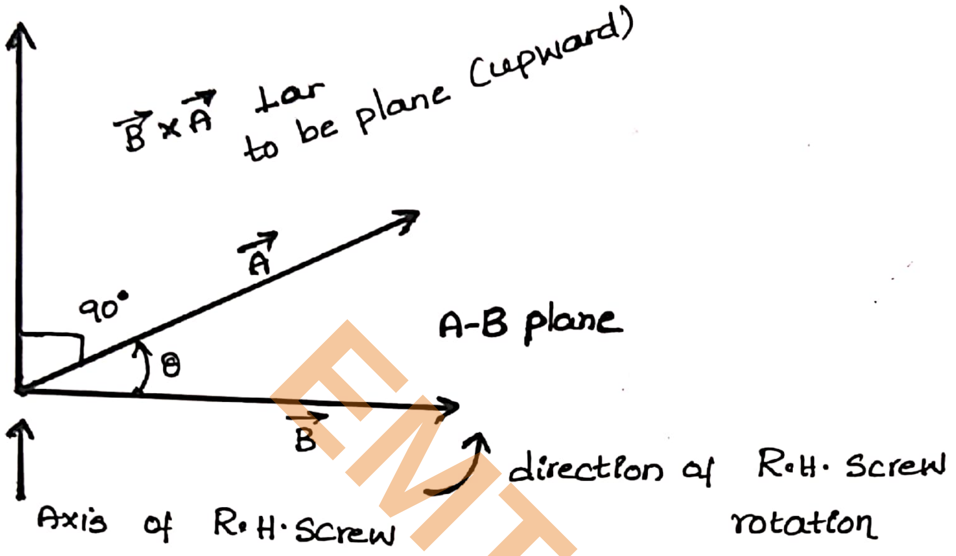
$\boxed{\vec{A}\times\vec{B} = |A||B|\sin\theta}$, Where 'θ' is the angle between $\vec{A}$ & $\vec{B}$

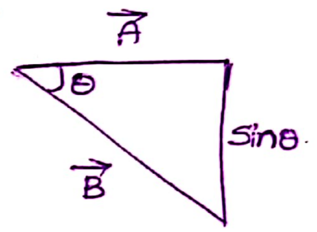
The vector product of the two vectors $\vec{A}$ and $\vec{B}$ is represented by putting cross (x) between the two Vectors and hence it is also called as cross product.

This relation can be computed by using right hand screw Rule

i.e the direction of advance of a right hand Screw whose axis is $\perp$ar to the plane of vector $\vec{A}$ and vector $\vec{B}$ and is rotated from first vector $\vec{A}$ in to 2nd vector $\vec{B}$ through the smaller angle.







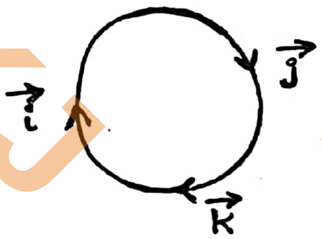
$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

Axis is downward (+ve sign) Axis is upward (-ve sign)

first touches $\vec{A}$ $\sin\theta$ $\vec{B}$ $= \vec{A}\,\vec{B}\sin\theta$

Relation between unit vectors using cross product

- According to cross product



case 1: $\vec{i} \times \vec{i} = |i||i| \sin\theta$

from i to i $\theta = 0°$

$$\therefore \boxed{\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = 0}$$

case 2: $\vec{i} \times \vec{j} = |i||j| \sin\theta$

from i to j $\theta = 90°$

The cross product of $\vec{i} \times \vec{j}$ involves 90° and hence the result is a third unit vector in the direction of right handed screw rotated from $\vec{i}$ into $\vec{j}$, which is in the direction of $\vec{k}$. Therefore

$$\left.\begin{aligned} \vec{i} \times \vec{j} &= \vec{k} \\ \vec{j} \times \vec{k} &= \vec{i} \\ \vec{k} \times \vec{i} &= \vec{j} \end{aligned}\right\} \qquad \left.\begin{aligned} \vec{j} \times \vec{i} &= -\vec{k} \\ \vec{k} \times \vec{j} &= -\vec{i} \\ \vec{i} \times \vec{k} &= -\vec{j} \end{aligned}\right\} \text{Reverse rotation}$$

In general, cross product of two like unit vectors is zero and of two unlike unit vectors is a third unit vector having positive sign for normal rotation and negative for reverse rotation.

Cross product of any two consecutive unit vectors

= Third unit vector with sign $\left[\begin{array}{l} +ve \text{ for clockwise cyclic rotation} \\ -ve \text{ for anti-clockwise cyclic rotation} \end{array}\right.$

$$\vec{A}\times\vec{B} = (\vec{i}A_x + \vec{j}A_y + \vec{k}A_z)\times(\vec{i}B_x + \vec{j}B_y + \vec{k}B_z)$$

$$= (\vec{i}\times\vec{i})A_xB_x + (\vec{i}\times\vec{j})A_xB_y + (\vec{i}\times\vec{k})A_xB_z$$

$$+ (\vec{j}\times\vec{i})A_yB_x + (\vec{j}\times\vec{j})A_yB_y + (\vec{j}\times\vec{k})A_yB_z$$

$$+ (\vec{k}\times\vec{i})A_zB_x + (\vec{k}\times\vec{j})A_zB_y + (\vec{k}\times\vec{k})A_zB_z$$

$$= \vec{k}(A_xB_y - A_yB_x) + \vec{i}(-A_zB_y + A_yB_z) + \vec{j}(A_zB_x - A_xB_z)$$

$$\boxed{\vec{A}\times\vec{B}} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Cross product does not obeys commutative law $i\times j \neq j\times i$

dot product obeys commutative law $i\cdot j = j\cdot i$

Unit Vector Representation :-

A vector say $\vec{A}$ whose magnitude is $|A|$. Let 'e' be a unit vector in the direction of $\vec{A}$. Symbolically it can be written that

$$\vec{A} = |A|\,e \rightarrow \text{unit vector in the direction of } \vec{A}$$

$\vec{A}$ ↓ Vector

$|A|$ ↓ Magnitude of vector $\vec{A}$

$$e = \frac{\vec{A}}{|A|} = \frac{\vec{i}A_x + \vec{j}A_y + \vec{k}A_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}}$$

if $A \neq 0$

Multiplication of three Vectors (or) Vector triple product

(i) $(\vec{A}\cdot\vec{B})\times\vec{C}$ (ii) $\vec{A}\cdot(\vec{B}\times\vec{C})$ (iii) $\vec{A}\times(\vec{B}\times\vec{C})$

(ii) $\vec{A}\cdot(\vec{B}\times\vec{C}) = (\vec{i}A_x+\vec{j}A_y+\vec{k}A_z)\cdot\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ B_x & B_y & B_z\\ C_x & C_y & C_z\end{vmatrix}$

$$(\vec{i}A_x+\vec{j}A_y+\vec{k}A_z)\cdot(\vec{i}(B_yC_z-B_zC_y)-\vec{j}(B_xC_z-B_zC_x)+\vec{k}(B_xC_y-B_yC_x))$$

$$A_x(B_yC_z-B_zC_y)-A_y[(B_xC_z)-B_zC_x)]+A_z(B_xC_y-B_yC_x)$$

$$\underline{\vec{A}\cdot(\vec{B}\times\vec{C})} = \begin{vmatrix}A_x & A_y & A_z\\ B_x & B_y & B_z\\ C_x & C_y & C_z\end{vmatrix}$$

IIIly

$$\underline{\vec{B}\cdot(\vec{C}\times\vec{A})} = \begin{vmatrix}B_x & B_y & B_z\\ C_x & C_y & C_z\\ A_x & A_y & A_z\end{vmatrix}$$

$$\underline{\vec{C}\cdot(\vec{A}\times\vec{B})} = \begin{vmatrix}C_x & C_y & C_z\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}$$

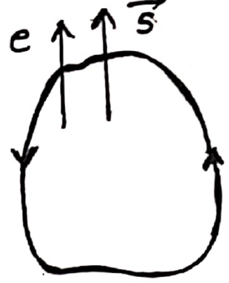
$$\boxed{\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})}$$ Verify

Problem 1: Proove that

$$\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})C$$

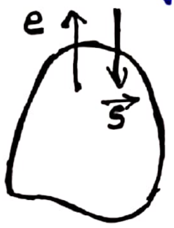
Associative Law does not hold good for vector triple product i.e $\boxed{\vec{A}\times(\vec{B}\times\vec{C}) \neq (\vec{A}\times\vec{B})\times\vec{C}}$

Vector Representation

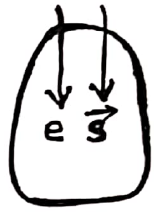


consider an arbitary plane surface. This surface may be represented by a vector say $\vec{s}$ and its unit vector is represented by a letter 'e'.

→ In the above fig. the vector direction and unit vector direction is outward. In this case the sign of the vector is considered as positive (+ sign)



→ negative Vector



→ -Sign Vector

Vector Analysis :-

Vector differential operator (∇) dell (or) Nabla is known as differential vector operator.

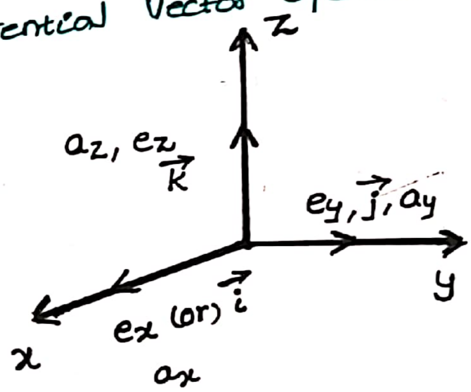
It is defined for three dimensional case

$$\nabla = \vec{i}\frac{d}{dx} + \vec{j}\frac{d}{dy} + \vec{k}\frac{d}{dz}$$

Del has units of [1/m]

$$= e_x\frac{d}{dx} + e_y\frac{d}{dy} + e_z\frac{d}{dz}$$

$$= a_x\frac{d}{dx} + a_y\frac{d}{dy} + a_z\frac{d}{dz}$$



different notations are used to represent the unit vectors.

→ ∇ is merely (nothing but) a vector operator and not a vector quantity

→ The properties of vector differential operator (∇) are similar to the ordinary vectors.

Del is operated in three ways

There are 3 possible operations of ∇

1> Gradient   2> Divergence   3> curl (or) rotation

1. Gradient :- 1st type of multiplication

If the vector differential operator (∇) is operated with a scalar quantity $\phi(x,y,z)$ the result is a vector.

$$\nabla\phi = \left(\vec{i}\frac{d}{dx} + \vec{j}\frac{d}{dy} + \vec{k}\frac{d}{dz}\right)\phi$$

$$= \vec{i}\frac{d\phi}{dx} + \vec{j}\frac{d\phi}{dy} + \vec{k}\frac{d\phi}{dz}$$

Thus gradient is a vector point function derived from a scalar point function. ex:- are gradient of temp, gradient of electric potential & so on.

2. Divergence :- 2nd type of multiplication

The vector differential operator '∇' is operated with a another vector say $\vec{A}$ then

$$\nabla\cdot\vec{A} = \left(\vec{i}\frac{d}{dx} + \vec{j}\frac{d}{dy} + \vec{k}\frac{d}{dz}\right)\cdot\left(\vec{i}A_x + \vec{j}A_y + \vec{k}A_z\right)$$

***

$$\boxed{\nabla\cdot\vec{A} = \frac{dA_x}{dx} + \frac{dA_y}{dy} + \frac{dA_z}{dz}}$$

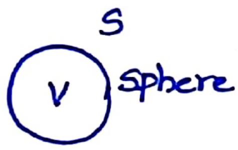
$\nabla\vec{A}$ = excess outward flow over inward flow i.e divergence of fluid

$-\nabla\cdot\vec{A}$ = excess inward flow over outward flow i.e convergence of fluid

Alternative definition of Divergence :-

The divergence of a vector field $\vec{A}$ mathematically can be defined as the limit of the net outward flux $\vec{A}$, $\oint_s \vec{A}\cdot\vec{ds}$ per unit volume $\vec{dv}$ enclosed by the surface 's' approaches towards zero. Symbolically

***



$$\boxed{\mathrm{Div}\ \vec{A} = \underset{dv\to 0}{Lt} \frac{\oint_s \vec{A}\cdot\vec{ds}}{dv}}$$

$$\nabla\cdot\vec{A} = \frac{dA_x}{dx} + \frac{dA_y}{dy} + \frac{dA_z}{dz} = \underset{dv\to 0}{Lt} \frac{\oint_s \vec{A}\cdot\vec{ds}}{dv}$$

3. curl (or) rotation :- 3rd type of multiplication

$$\nabla\times\vec{A} = \left(\vec{i}\frac{d}{dx} + \vec{j}\frac{d}{dy} + \vec{k}\frac{d}{dz}\right)\times\left(\vec{i}A_x + \vec{j}A_y + \vec{k}A_z\right)$$

$$= \vec{i}\left[\frac{dA_z}{dy} - \frac{dA_y}{dz}\right] + \vec{j}\left[\frac{dA_x}{dz} - \frac{dA_z}{dx}\right] + \vec{k}\left[\frac{dA_y}{dx} - \frac{dA_x}{dy}\right]$$

$$\nabla \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ A_x & A_y & A_z \end{vmatrix}$$

Alternative definition of Curl :

The component of the curl of a vector normal to a surface 'S' is equal to the line integral of the vector around the boundary of the surface divided by the area of the surface for the limiting case in which the surface area tends to zero.

Symbolically

$$\boxed{\nabla \times \vec{A} = \underset{S \to 0}{\mathrm{lt}} \frac{\oint_L \vec{A} \cdot \vec{dl}}{S}}$$

Line integral :- is defined as any integral which is to be evaluated along a curve.

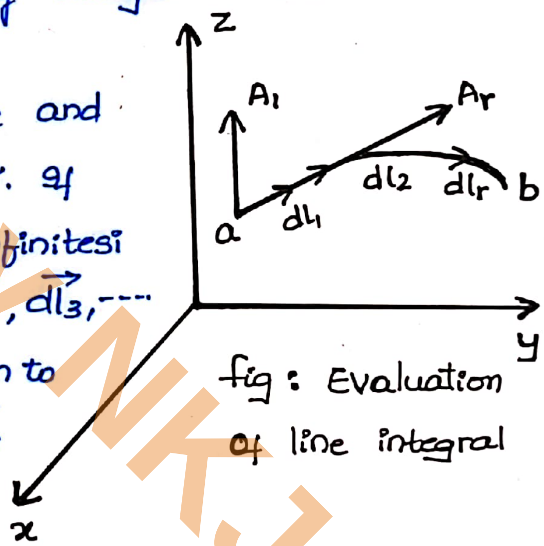


fig : Evaluation of line integral

Suppose $\vec{A}$ be a vector field in space and ab a curve from point 'a' to point 'b'. If the curve ab is subdivided into infinitesimally small vector elements $\vec{dl_1}, \vec{dl_2}, \vec{dl_3}, \ldots \vec{dl_r}$ and the vector $\vec{A}$ is divided into small vector elements $\vec{A_1}, \vec{A_2}, \vec{A_3}, \ldots \vec{A_r}$ and the scalar products $\vec{A_1} \cdot \vec{dl_1}$, $\vec{A_2} \cdot \vec{dl_2}$, $\vec{A_3} \cdot \vec{dl_3}, \ldots \vec{A_r} \cdot \vec{dl_r}$ are taken.

$\longrightarrow$ one dimension

The sum of these products i.e

$$= \vec{A_1} \cdot \vec{dl_1} + \vec{A_2} \cdot \vec{dl_2} + \ldots + \vec{A_r} \cdot \vec{dl_r}$$

$$= \sum_a^b \vec{A_r} \cdot \vec{dl_r} = \int_a^b \vec{A} \cdot \vec{dl}$$ along the entire length

$\downarrow$ discrete domain $\qquad$ $\hookrightarrow$ continuous domain

of the curve is known as the line integral of $\vec{A}$ along the curve ab.

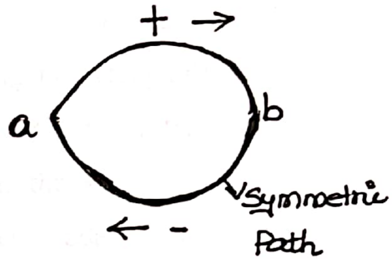
If the line integral around the closed path is zero.

$$\boxed{\oint \vec{A} \cdot \vec{dl} = 0}$$

(: The circle on the integral sign indicates that the path is closed)

Line integral is used to measure the area of the line.

<u>Note</u>: Line integral from point 'b' to point 'a' will be negative to that from point 'a' to point 'b'.



<u>Surface integral</u> :-

Let upper side of surface 's' be considered as positive side arbitrarily.

$\vec{e}$ be unit vector normal to any point of the positive side

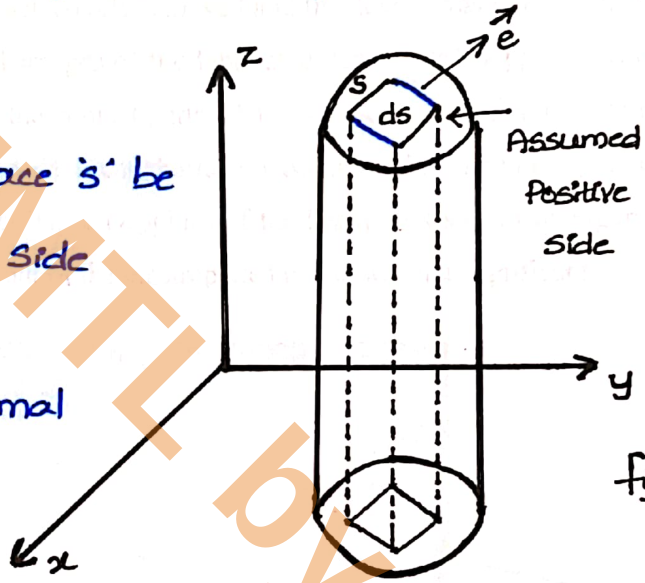


fig : A two sided Surface

closed path defines open surface

ds = differential surface area

$\vec{ds}$ = Vector surface area having magnitude equal to ds and direction, in the direction of unit normal $\vec{e}$. This can be written as

$$\vec{ds} = ds\vec{e}$$

Now taking the dot product with vector $\vec{A}$ and integrating (twice integrals because of two sides of surface 's')

$$\psi = \iint_S \vec{A} \cdot \vec{ds} \qquad \boxed{\iint_S \vec{A} \cdot \vec{ds} = \iint_S \vec{A} \cdot \vec{e}\, ds}$$

Surface integral is also called as the <u>flux of $\vec{A}$ over surface</u> 's'.

Surface integral in terms of flux is represented as

$\boxed{flux \rightarrow \phi}$ $\qquad \iint_S \phi \cdot \vec{ds}$

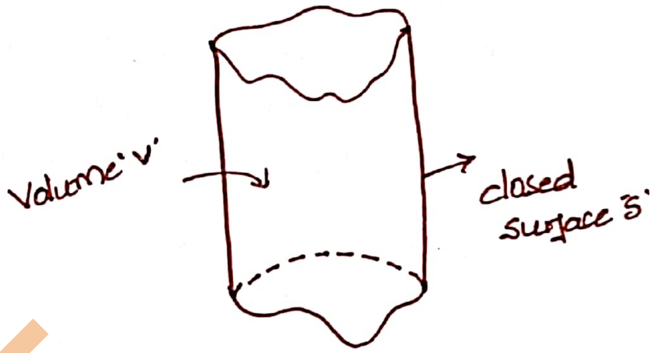
consider given a vector field 'A', continuous in a region containing the smooth surface 'S' we define the surface integral (or the flux of 'A' through 's' as

For a closed Surface. the Surface integral of above becomes

$$\psi = \oint_S A \cdot ds$$

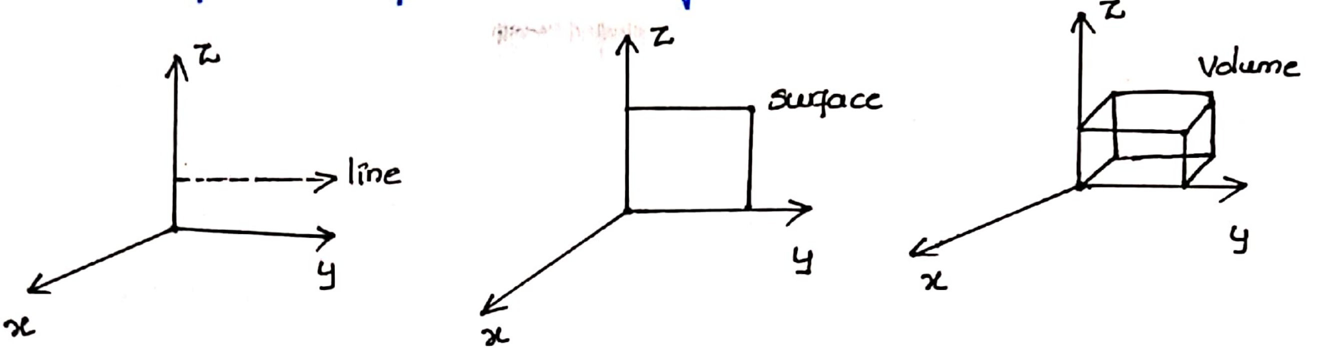
Which is referred to as the net outward flux of 'A' from 'S'.

NOTE : closed Path defines an open Surface where as a closed Surface defines a Volume.



Volume integral :-

If a closed surface in space enclosing a volume 'V' is considered



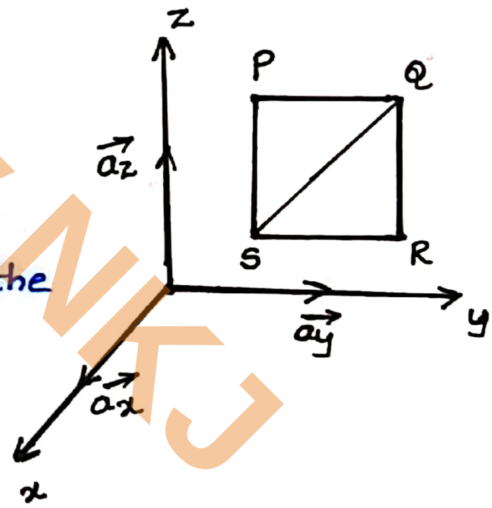
(a) Line integral → one dimension → $\int_L \vec{A}\cdot\vec{dl}$

(b) Surface integral → two dimensional → $\iint_S \vec{A}\cdot\vec{ds}$

(c) Volume integral → three dimensional → $\iiint_V \vec{A}\cdot\vec{dv}$

Differential Surface area & Differential length

1> differential length



differential length from P to Q (or) Q to P is $\boxed{dy\,\vec{a_y}}$ because we are moving in the y-directional $\vec{a_y}$ is the unit vector along the y-direction

From P to Q $\quad \vec{dl} = dy\,\vec{a_y}$

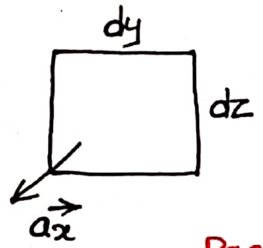
Q to R $\quad \vec{dl} = dz\,\vec{a_z}$

Q to S $\quad \vec{dl} = dx\,\vec{a_x}$

2> differential Surface area (or) differential normal area

$$\boxed{\vec{ds} = ds\,\vec{a_n}}$$

Where ds is the area of the surface element and $\vec{a_n}$ is a unit vector normal to the surface ds.



here y & z forms the surface and $\vec{a_x}$ is a unit vector normal to the surface

$\vec{ds} = dydz\,\vec{a_x}$ → differential surface area along $\vec{a_x}$ direction.

$= dxdz\,\vec{a_y}$

$= dxdy\,\vec{a_z}$.

(or) $ds_x = dydz$
$ds_y = dxdz$
$ds_z = dxdy$

Integral theorems :-

There are two most important types of integral theorems

1 > Divergence theorem (or) Gauss's theorem

2 > Stroke's theorem.

1. Gauss's Divergence theorem :

It permits us to express Certain integrals by means of surface integrals.

The divergence theorem relates to a closed volume 'V' in space and the surface that bounds it.

Statement :- States that "the Volume integral of the divergence of a Vector field $\vec{A}$ taken over any volume $\vec{V}$ is equal to the surface integral of $\vec{A}$ taken over any closed surface Surrounding the Volume $\vec{V}$."

***

$$\boxed{\iiint_V (\nabla\cdot\vec{A})\,\vec{dV} = \iint_S \vec{A}\cdot\vec{ds} = \iint_S \vec{A}\cdot\vec{e}\,ds}$$

$\int_V \nabla\cdot A\,dv = \oint_S A\cdot ds$

Proof :

L.H.S ⇒ $\nabla\cdot\vec{A} = \frac{dA_x}{dx} + \frac{dA_y}{dy} + \frac{dA_z}{dz}$

$\vec{dV} = dxdydz$

$$\therefore \iiint_V \left(\frac{dA_x}{dx} + \frac{dA_y}{dy} + \frac{dA_z}{dz}\right) dxdydz$$

$$= \iiint_V \frac{dA_x}{dx}\,dxdydz + \iiint_V \frac{dA_y}{dy}\,dxdydz + \iiint_V \frac{dA_z}{dz}\,dxdydz$$

consider the first integral on R.H.side

$$\iiint_V \frac{dA_x}{dx}\,dxdydz = \iint\left[\int \frac{dA_x}{dx}\,dx\right]dydz$$

... vectors $\vec{a}$ from

$$= \iint_S [A_x]\, dy\, dz = \iint_S [A_x][ds_x]$$

|||ly

$$\iint_S [A_y]\, dx\, dz = \iint_S [A_y][ds_y]$$

$$\iint_S [A_z]\, dx\, dy = \iint_S [A_z][ds_z]$$

$ds_x \rightarrow$ differential surface area along $\vec{a}_x$ direction.

$$\therefore \iint_S [A_x][ds_x] + \iint_S [A_y][ds_y] + \iint_S [A_z][ds_z]$$

$$= \iint_S \vec{A}\cdot\vec{ds}$$

$\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z$

hence

$$\boxed{\iiint_V (\nabla\cdot\vec{A})\, \vec{dv} = \iint_S \vec{A}\cdot\vec{ds}}$$ prooved

Stroke's Theorem :- This theorem permits us to transform certain integrals caluculated over the surface in to line integrals. Transforms line integrals in to surface integrals.

Statement :- States that the surface integral of the curl of a vector field $\vec{A}$ taken over any surface 's' is equal to the line integral of $\vec{A}$ around the closed contour of the surface. (or)

The line integral of the tangential component of a vector $\vec{A}$ around the closed path is equal to the surface integral of the normal component of $\nabla\times\vec{A}$ over the surface enclosed by the path.

$$\boxed{\iint_S (\nabla\times\vec{A})\cdot\vec{ds} = \oint_L \vec{A}\cdot\vec{dl} = \iint_S (\nabla\times\vec{A})\cdot\vec{e}\, ds}$$

where 'L' is the closed contour bounding the surface 's'.

Proof :- consider an arbitrary surface 's' as shown in fig. and divide the surface in to differential elemental areas $ds_1, ds_2, \ldots ds_n$ etc.

$$\int_S (\nabla\times A)\cdot ds = \oint_L A\cdot dL$$

For each such elementary area line integral $\oint \vec{A}\cdot\vec{dl}$ will be taken in the positive sense (or) anti clockwise direction and on summing all these integrals over the surface the only integral remains after summing is the original boundary enclosing L.

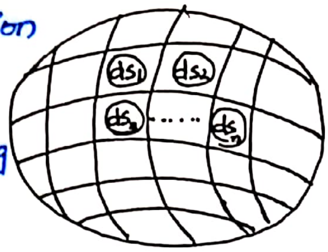


fig: Proof of Stroke's theorem.

$$\oint_L \vec{A}\cdot\vec{dl} = \oint_{ds_1} \vec{A}\cdot\vec{dl} + \oint_{ds_2} \vec{A}\cdot\vec{dl} + \oint_{ds_3} \vec{A}\cdot\vec{dl} + \cdots\cdots + \oint_{ds_n} \vec{A}\cdot\vec{dl} \quad \text{—} \quad (1)$$

Alternative def of curl :

$$\nabla\times\vec{A} = \underset{S\to 0}{\text{lt}} \frac{\oint_L \vec{A}\cdot\vec{dl}}{S}$$

(or)

$$\nabla\times\vec{A} = \underset{ds\to 0}{\text{lt}} \frac{\oint_L \vec{A}\cdot\vec{dl}}{\vec{ds}}$$ for elemental area ds

with in limit it may be written as

$$\oint_L \vec{A}\cdot\vec{dl} = (\nabla\times\vec{A})\cdot\vec{ds} \quad \text{—} \quad (2)$$

Substitute eqn (2) in to eqn (1)

$$\oint_L \vec{A}\cdot\vec{dl} = (\nabla\times\vec{A})\cdot\vec{ds_1} + (\nabla\times\vec{A})\cdot\vec{ds_2} + (\nabla\times\vec{A})\cdot\vec{ds_3} + \cdots\cdots$$

$$= \iint_S (\nabla\times\vec{A})\cdot\vec{ds} \quad \text{—} \quad (3)$$

R.H.S side of eqn (3) indicates the summation of normal components of $\nabla\times\vec{A}$ over the entire surface and hence R.H.S is written as surface integral of the normal component of $\nabla\times\vec{A}$

***

$$\boxed{\oint_L \vec{A}\cdot\vec{dl} = \iint_S (\nabla\times\vec{A})\cdot\vec{ds}}$$

hence Stroke's theorem is prooved.

Co-ordinate system :-

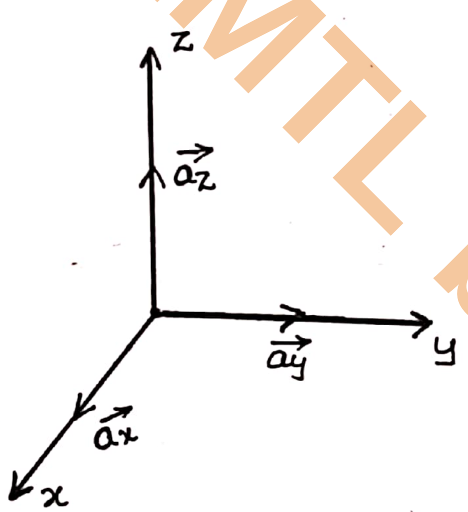
→ Some specific Lengths, directions, Projections, angles (or) Components are required to describe a Vector accurately. For this Purpose three methods are used.

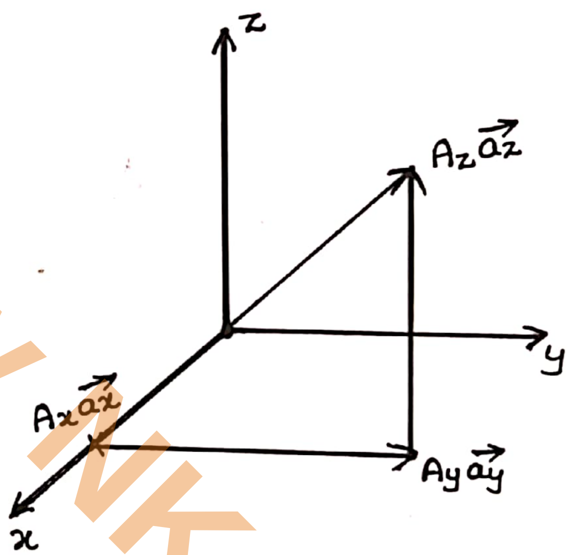
→ A Point (or) Vector can be represented in any co-ordinate system which may be orthogonal (or) non-orthogonal.

→ An orthogonal system is one in which the co-ordinates are mutually perpendicular.

Ex :- Cartesian ((or) rectangular), cylindrical & Spherical co-ordinate system.

Cartesian (or) Rectangular Co-ordinates (x, y, z)





→ $\vec{a_x}$, $\vec{a_y}$ & $\vec{a_z}$ are the unit vectors along the x, y and z directions

→ The ranges of the co-ordinate variables x, y and z are

$$-\infty \le x \le \infty \ ; \ -\infty \le y \le \infty \ ; \ -\infty \le z \le \infty$$

→ A Vector in Cartesian Coordinates can be written as

(A_x, A_y, A_z) (or) $\vec{A} = A_x\vec{a_x} + A_y\vec{a_y} + A_z\vec{a_z}$

Circular cylindrical coordinate system (ρ, ϕ, z)

→ It is a very Convenient co-ordinate system whenever we are dealing with Problems having cylindrical Symmetry

→ A point 'P' in cylindrical co-ordinates is represented as (ρ, ϕ, z) as shown in fig.

→ ρ is the radius of the cylinder passing through the point 'P' (or) radial distance from the z-axis.

→ ϕ called the azimuth angle is measured from the x-axis in the x-y plane and z is same as in the cartesian system.

→ The ranges of the variables are

$0 \le \rho \le \infty$; $0 \le \phi \le 2\pi$

$-\infty \le z \le \infty$

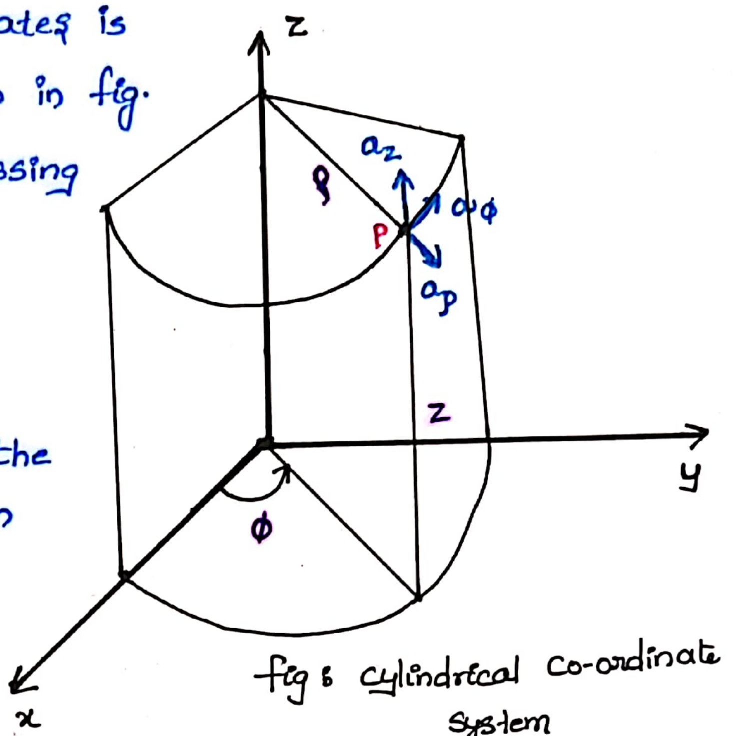


fig: cylindrical co-ordinate system

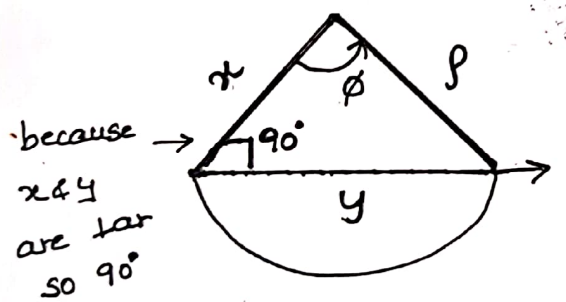
→ The vector $\vec{A}$ in cylindrical co-ordinates can be written as

(A_ρ, A_ϕ, A_z) (or) $A_\rho \vec{a_\rho} + A_\phi \vec{a_\phi} + A_z \vec{a_z} = \vec{A}$

$\vec{a_\rho}, \vec{a_\phi}, \vec{a_z}$ are the unit vectors in the directions of ρ, ϕ & z

magnitude of $\vec{A}$ = $|A| = \sqrt{A_\rho^2 + A_\phi^2 + A_z^2}$

Consider lower part of fig 1:



$\sin\phi = \boxed{\dfrac{y}{\rho}}$ 90° opposite - hypotenus

$y = \rho \sin\phi$

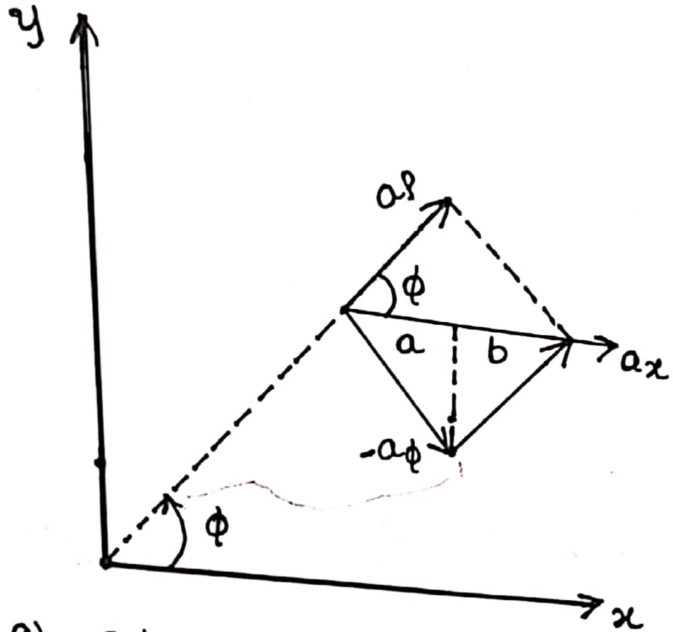
$\cos\phi = x/\rho \Rightarrow x = \rho\cos\phi$

$z = z$

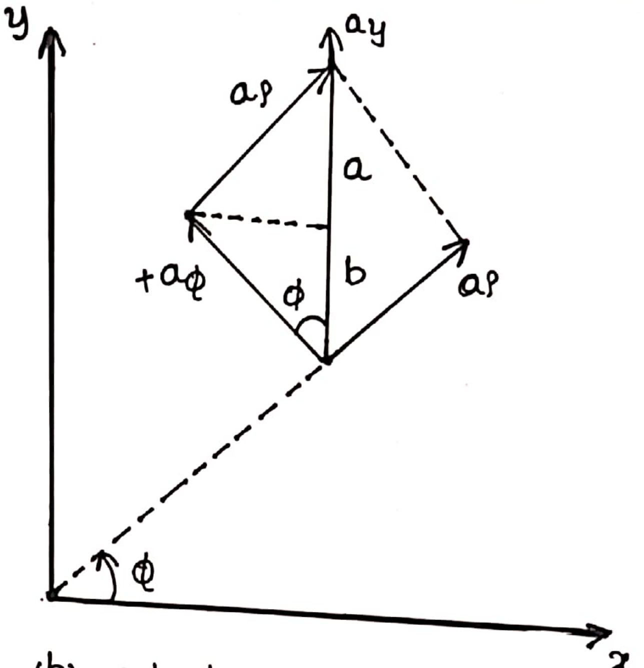
Relation between cartesian & cylindrical co-ordinates is

$x = \rho\cos\phi$	$\rho = \sqrt{x^2+y^2}$
$y = \rho\sin\phi$	$\phi = \tan^{-1}(x/y)$
$z = z$	$= \tan^{-1}(y/x)$

...ation of Vectors $\vec{A}$ from



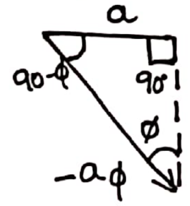
a) Cylindrical components of a_x



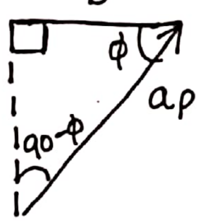
(b) cylindrical components of a_y

fig : Unit Vector transformation

from fig (a)



$\sin\phi = \dfrac{a}{-a_\phi} \Rightarrow a = -a_\phi \sin\phi$

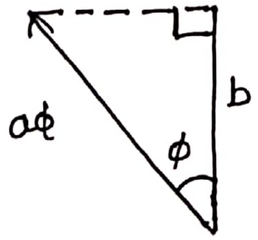


$\cos\phi = \dfrac{b}{a_\rho} \Rightarrow b = a_\rho \cos\phi$

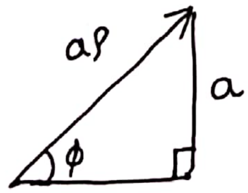
$$a_x = a + b$$

$$\boxed{a_x = -a_\phi \sin\phi + a_\rho \cos\phi}$$

from fig (b)



$\cos\phi = \dfrac{b}{a_\phi} \Rightarrow b = a_\phi \cos\phi$



$\sin\phi = \dfrac{a}{a_\rho} \Rightarrow a = a_\rho \sin\phi$

$$a_y = a + b$$

$$\boxed{a_y = a_\rho \sin\phi + a_\phi \cos\phi}$$

$$a_z = a_z$$

$$\left.\begin{aligned}\vec{a_x} &= \vec{a_\rho}\cos\phi - \vec{a_\phi}\sin\phi \\ \vec{a_y} &= \vec{a_\rho}\sin\phi + \vec{a_\phi}\cos\phi \\ \vec{a_z} &= \vec{a_z}\end{aligned}\right\}$$ Relation between unit vectors in Cartesian & cylindrical co-ordinates

The vector $\vec{A}$ in Cartesian co-ordinate system is represented as

$$\vec{A} = A_x\vec{a_x} + A_y\vec{a_y} + A_z\vec{a_z}$$

$$\vec{A} = A_x(\vec{a_\rho}\cos\phi - \vec{a_\phi}\sin\phi) + A_y(\vec{a_\rho}\sin\phi + \vec{a_\phi}\cos\phi) + A_z\vec{a_z}$$

$$= \vec{a_\rho}(A_x\cos\phi + A_y\sin\phi) + \vec{a_\phi}(A_y\cos\phi - A_x\sin\phi) + A_z\vec{a_z} \quad \text{--- (1)}$$

$\vec{A}$ in cylindrical co-ordinate system

$$\vec{A} = A_\rho\vec{a_\rho} + A_\phi\vec{a_\phi} + A_z\vec{a_z} \quad \text{--- (2)}$$

compare eqn's (1) & (2) we get

$$A_\rho = A_x\cos\phi + A_y\sin\phi \; ; \; A_\phi = A_y\cos\phi - A_x\sin\phi \; ; \; A_z = A_z$$

$$\vec{a_x}\sin\phi - \vec{a_y}\cos\phi = \vec{a_\rho}\cancel{\cos\phi\sin\phi} - \vec{a_\phi}\sin^2\phi - \vec{a_\rho}\cancel{\cos\phi\sin\phi} + (-\vec{a_\phi}\cos^2\phi)$$

$$= -\vec{a_\phi}(\sin^2\phi + \cos^2\phi)$$

$$\left.\begin{aligned}\Rightarrow \quad \vec{a_\phi} &= -\vec{a_x}\sin\phi + \vec{a_y}\cos\phi \\ \text{|||}^{ly} \quad \vec{a_\rho} &= \vec{a_x}\cos\phi + \vec{a_y}\sin\phi \\ \vec{a_z} &= \vec{a_z}\end{aligned}\right\}$$

Dot Products of unit vectors in cylindrical & cartesian co-ordinate system

	a_ρ	a_ϕ	a_z
$a_x \cdot$	$\cos\phi$	$-\sin\phi$	0
$a_y \cdot$	$\sin\phi$	$\cos\phi$	0
$a_z \cdot$	0	0	1

In matrix form the transformation of vectors $\vec{A}$ from (A_x, A_y, A_z) to (A_ρ, A_ϕ, A_z) as

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

Spherical co-ordinate system (r, θ, ϕ) :-

→ 'r' is defined as the distance from the origin to point 'P' (or the radius of a sphere centered at the origin and passing through 'P'.

→ 'θ' is the angle between the z-axis and the position vector of P. (θ also called the colatitude)

→ ϕ is measured from the x-axis (same as the azimuthal angle in cylindrical co-ordinates)

A point 'P' can be represented as (r, θ, ϕ)

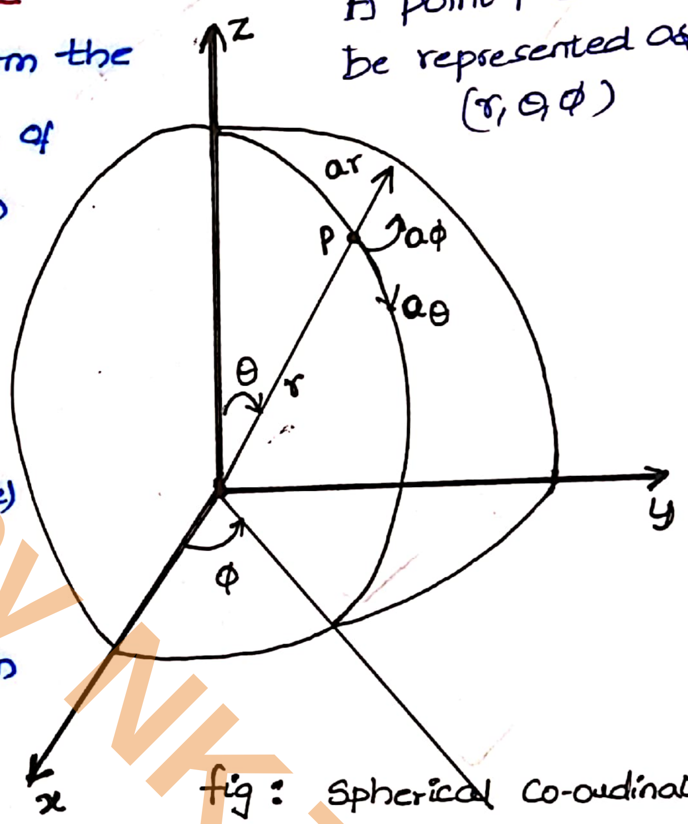


fig : Spherical co-ordinate system

→ The ranges of the variables are

$$0 \le r \le \infty \; ; \quad 0 \le \theta \le \pi \; ; \quad 0 \le \phi \le 2\pi$$

A vector $\vec{A}$ in spherical co-ordinates may be written as (A_r, A_θ, A_ϕ) (or) $\vec{A} = A_r \vec{a_r} + A_\theta \vec{a_\theta} + A_\phi \vec{a_\phi}$

$$|A| = (A_r^2 + A_\theta^2 + A_\phi^2)^{1/2}$$

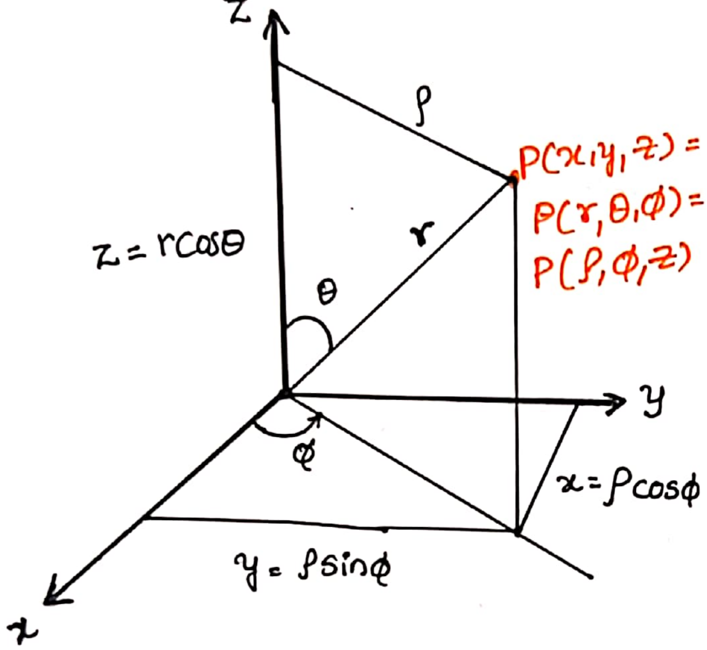
Lower part is same as cylindrical co-ordinates so

$x = \rho\cos\phi$

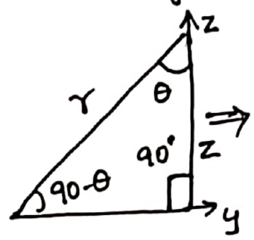
$y = \rho\sin\phi$

$z = z$

fig: Relationship between space variables (x,y,z), (r,θ,ϕ) & (ρ,ϕ,z)

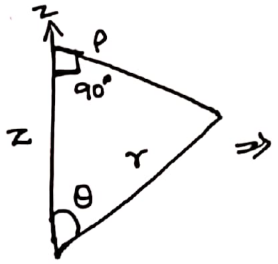


from fig



$\Rightarrow \quad \cos\theta = \dfrac{z}{r}$

$z = r\cos\theta$



$\Rightarrow \quad \sin\theta = \dfrac{\rho}{r}$

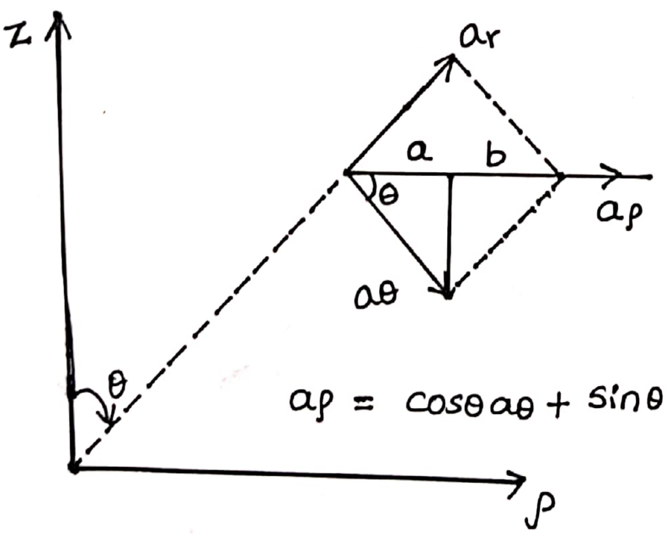
$\rho = r\sin\theta$

$$\left.\begin{aligned} x &= \rho\cos\phi = r\sin\theta\cos\phi \\ y &= \rho\sin\phi = r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}\right\} \quad \begin{aligned} r &= \sqrt{x^2+y^2+z^2} \\ \theta &= \tan^{-1}\sqrt{\frac{x^2+y^2}{z^2}} \end{aligned}$$

$$x^2+y^2+z^2 = (r\sin\theta\cos\phi)^2 + (r\sin\theta\sin\phi)^2 + (r\cos\theta)^2$$

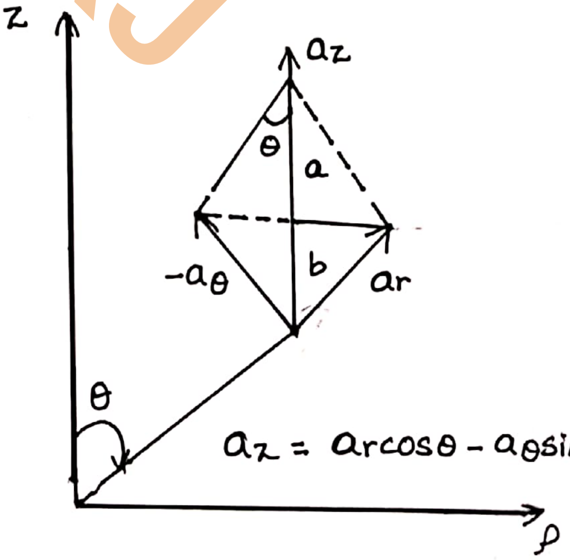
$$= r^2 \Rightarrow r = \sqrt{x^2+y^2+z^2}$$

$$\frac{x^2+y^2}{z^2} = \frac{r^2\sin^2\theta}{r^2\cos^2\theta} \Rightarrow \tan^2\theta = \frac{x^2+y^2}{z^2} \Rightarrow \theta = \tan^{-1}\sqrt{\frac{x^2+y^2}{z^2}}$$



$a_\rho = \cos\theta\, a_\theta + \sin\theta\, a_r$

Spherical components of a_ρ



$a_z = a_r\cos\theta - a_\theta\sin\theta$

fig: unit vector transformation

$$\left.\begin{array}{l} a_\rho = \cos\theta\, a_\theta + a_r \sin\theta \\ a_z = a_r\cos\theta - a_\theta \sin\theta \end{array}\right\} \quad a_\phi = a_\phi$$

$$a_\phi = -a_x \sin\phi + a_y \cos\phi$$

$$\left.\begin{array}{l} a_x = \cos\phi\, a_\rho - \sin\phi\, a_\phi \\ a_y = \sin\phi\, a_\rho + \cos\phi\, a_\phi \end{array}\right\} \quad \text{— } (1)$$

$$a_z = a_z$$

Substitute a_ρ and a_ϕ & a_z in the above eqn's

$$\left.\begin{array}{l} \vec{a_x} = \cos\phi\,(\cos\theta\,\vec{a_\theta} + \vec{a_r}\sin\theta) - \sin\phi\,(\vec{a_\phi}) \\ \quad\;\; = \vec{a_r}\sin\theta\cos\phi + \vec{a_\theta}\cos\theta\cos\phi - \sin\phi\,(\vec{a_\phi}) \\ \vec{a_y} = \vec{a_r}\sin\theta\sin\phi + \cos\theta\sin\phi\,\vec{a_\theta} + \vec{a_\phi}\cos\phi \\ \vec{a_z} = \vec{a_r}\cos\theta - \vec{a_\theta}\sin\theta \end{array}\right\} \begin{array}{l}\text{relations between} \\ \text{unit vectors in} \\ \text{Cartesian \& Spherical}\end{array}$$

|||ly

$$\vec{a_r} = \vec{a_x}\sin\theta\cos\phi + \vec{a_y}\sin\theta\sin\phi + \vec{a_z}\cos\theta$$

$$\vec{a_\theta} = \vec{a_x}\cos\theta\cos\phi + \vec{a_y}\cos\theta\sin\phi - \vec{a_z}\sin\theta$$

$$\vec{a_\phi} = \vec{a_x}(-\sin\phi) + \vec{a_y}\cos\phi$$

from the dot Product we get $\vec{a_r}$, $\vec{a_\theta}$ & $\vec{a_\phi}$

	a_r	a_θ	a_ϕ
$a_x \cdot$	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$a_y \cdot$	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$a_z \cdot$	$\cos\theta$	$-\sin\theta$	0

Table : dot products of unit vectors in spherical & cartesian co-ordinate system.

Differential length, Area and volume :-

Cartesian Co-ordinates :

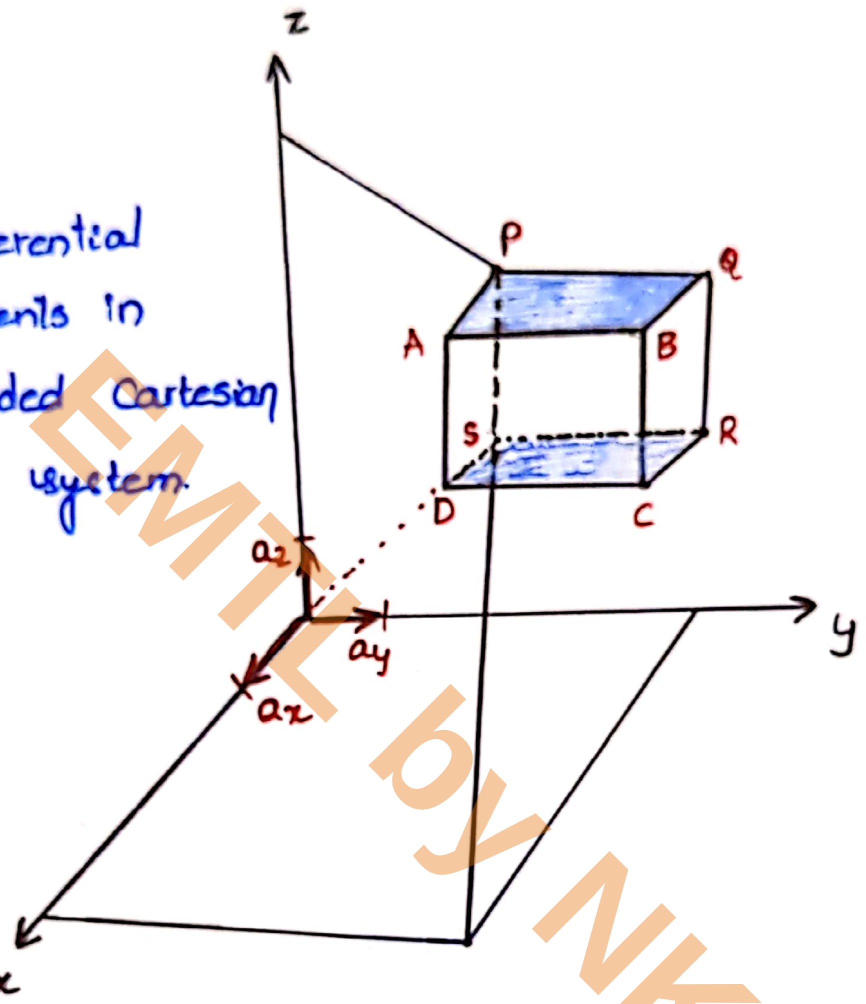


fig: Differential elements in right handed Cartesian Co-ordinate system.

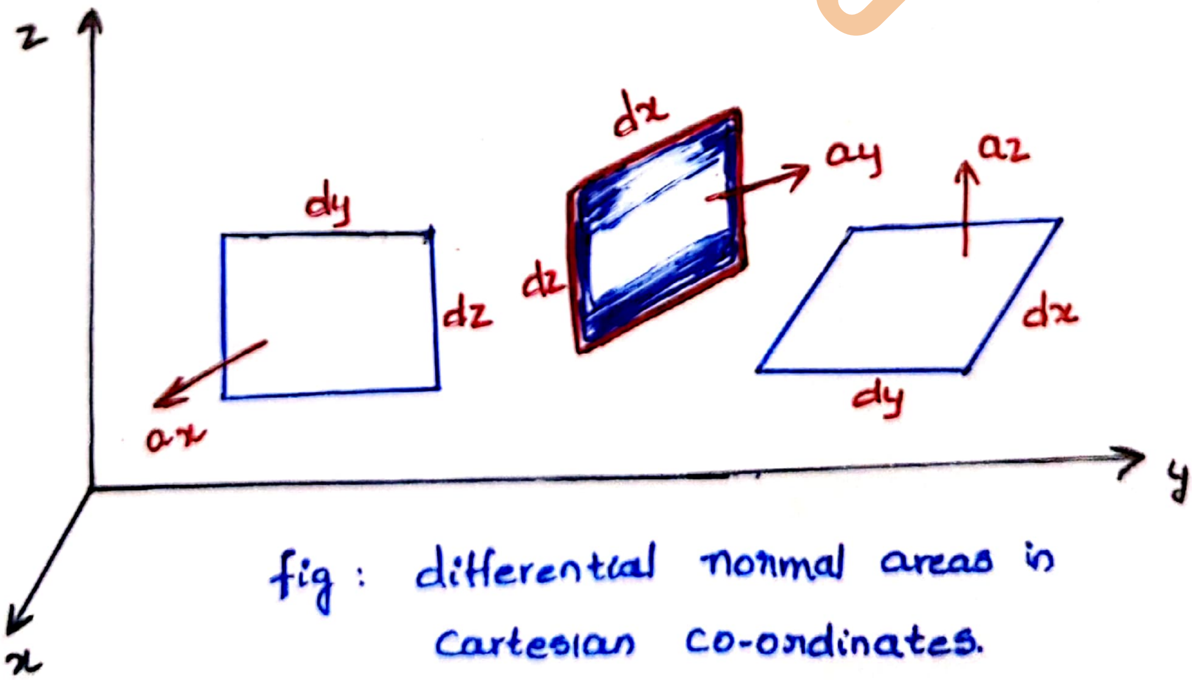


fig: differential normal areas in Cartesian co-ordinates.

Divergence of Vector (A) :-

$A = x^2 a_x + 2z a_y$

Divergence of vector function 'A' at a given point 'P' is the total flux per unit volume as the volume shrinks to 'P'.

$$\text{div } A = \nabla \cdot A = \lim_{\Delta v \to 0} \frac{\oint A \cdot ds}{\Delta v}$$

Where Δv is the volume enclosed by the closed surface 'S' in which 'P' is Located.

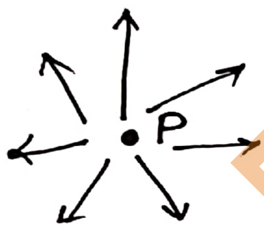


fig : 1

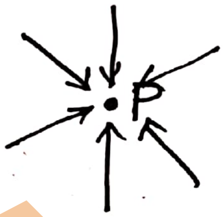


fig : 2

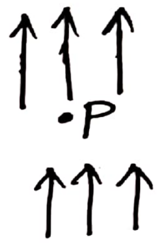


fig : 3

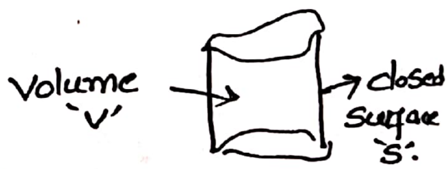
fig : 1 Shows that the divergence of a vector field at point 'P' is positive because the vector diverges (or spreads out) at point 'P'.

fig 2 : A vector field has negative divergence (or convergence) at 'P'.

fig 3 : A vector field has zero divergence at 'P'.

The physical meaning of divergence is how much the field diverges (or) emanates from a given point.

Divergence theorem :- [Surface ↔ Volume] applies to any volume 'V' bounded by a closed surface 'S'



States that the total flux is equal to the volume integral of divergence of any vector field.

$$\psi = \oint_S A \cdot ds = \int_v (\nabla \cdot A) \, dv.$$

$(x, y, z) \rightarrow$ Rectangular Co-ordinate system

$$\nabla \cdot A = \left[\frac{\partial}{\partial x} a_x + \frac{\partial}{\partial y} a_y + \frac{\partial}{\partial z} a_z\right] \cdot \left[A_x a_x + A_y a_y + A_z a_z\right]$$

$$= \left[\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right]$$

III^{ly} for cylindrical

$$\nabla \cdot A = \frac{1}{\rho} \frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

for spherical

$$\nabla \cdot A = \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta} \frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta} \frac{\partial A_\phi}{\partial \phi}.$$

Note :-

***

→ the divergence of curl of any vector is zero

$$\boxed{\nabla \cdot (\nabla \times A) = 0}$$

→ curl of gradient of any scalar is zero

$$\boxed{\nabla \times \nabla V = 0}$$

→ $\nabla \cdot (\nabla V) = \nabla^2 V$

Laplacian of a scalar :

To introduce a single operator which is the composite of gradient and divergence operators. This operator is known as the Laplacian

The Laplacian of a scalar field V, written as $\nabla^2 V$, is the divergence of the gradient of 'V'.

$$\text{Laplacian } V = \nabla \cdot \nabla V = \nabla^2 V.$$

$(x, y, z) \rightarrow$ cartesian co-ordinates

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

Divergence of a vector :- Divergence of a vector, A $(= \nabla \cdot A)$

→ Divergence means the spreading (or) diverging of a quantity from a point.

→ It is applicable to vectors only

→ The divergence of a vector indicates the net flow of quantities like gas, fluid, vapour, electric and magnetic flux lines.

→ In other words it is a measure of the difference between outflow and inflow.

→ The divergence of a vector is positive if the net flow is outward. It is negative if the net flow is inward.

→ The fluid is said to be incompressible if the divergence is zero. i.e $\nabla \cdot A$ is the condition of incompressibility.

Examples :

1 → Leaking air from a baloon yeilds positive divergence

2 • Rushing of air into the drum under the carriage of a train yields negative divergence.

3 • Divergence of water (or) oil is almost zero and hence they are incompressible.

Curl of a vector $(\equiv \nabla \times A)$

It is a measure of the tendency of a vector quantity to rotate (or) twist (or) curl. In other words, the rate of rotation (or) angular velocity at a point is the measure of curl.

As the curl of a vector represents rotation, it is also written as

$$\boxed{\text{curl } A = \text{rot } A = \nabla \times A}$$

Note :- curl of gradient of a scalar $= \nabla \times (\nabla V)$ is zero. this means that the gradient of fields is irrotational. also

$\text{div}(\text{curl}) = 0.$

Ex :- When a leaf floats in sea water and its rotation is about the z-axis, curl of velocity 'V' is in the z-direction. When $(\nabla \times V)_z$ is +ve, it represents rotation from 'x' to 'y'.

* Stroke's theorem gives the relation between a closed line integral and a surface integral.
* Divergence theorem gives the relation between a closed surface integral and a volume integral.
* ∇^2 is laplacian scalar operator.

Problems :-

① Express the following Vectors in Cartesian co-ordinates.

a> $\vec{A} = \rho z \sin\phi \, \vec{a_\rho} + 3\rho\cos\phi \, \vec{a_\phi} + \rho\cos\phi\sin\phi \, \vec{a_z}$

b> $\vec{B} = r^2 \vec{a_r} + \vec{a_\phi} \sin\theta$

Ans :

$$\vec{A} = \rho z \sin\phi \, \vec{a_\rho} + 3\rho\cos\phi \, \vec{a_\phi} + \rho\cos\phi\sin\phi \, \vec{a_z}$$

$$\rho = \sqrt{x^2+y^2} \, ; \quad \phi = \tan^{-1}(y/x) \, ; \quad \sin\phi = \frac{y}{\sqrt{x^2+y^2}}$$

$$\cos\phi = \frac{x}{\sqrt{x^2+y^2}}$$

$$\vec{A} = \left(\sqrt{x^2+y^2}\right)((z)) \frac{y}{\sqrt{x^2+y^2}} \left(\cos\phi \, \vec{a_x} + \sin\phi \, \vec{a_y}\right) + 3 \times \sqrt{x^2+y^2}$$

$$\times \frac{x}{\sqrt{x^2+y^2}} \left(-\sin\phi \, \vec{a_x} + \cos\phi \, \vec{a_y}\right) + \frac{xy}{\sqrt{x^2+y^2}} \vec{a_z}$$

$$= \frac{1}{\sqrt{x^2+y^2}} \left[\vec{a_x}\,(xyz - 3xy) + \vec{a_y}\,(zy^2 + 3x^2) + \vec{a_z}\, xy \right]$$

ⓑ $\vec{B} = r^2 \vec{a_r} + \vec{a_\phi} \sin\theta$

$$= \frac{1}{\sqrt{x^2+y^2+z^2}} \left[\left(x(x^2+y^2+z^2) - y\right) \vec{a_x} + \left[y\,(x^2+y^2+z^2) + x\right] \vec{a_y} + z(x^2+y^2+z^2)\, \vec{a_z} \right]$$

Next problem

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ x+z \\ 0 \end{bmatrix}$$

$A_x = y$

$A_y = x+z$

$A_z = 0$

$$\vec{A} = A_\rho \vec{a_\rho} + A_\phi \vec{a_\phi} + A_z \vec{a_z}$$

2> Given point $P(-2,6,3)$ and Vector $\vec{A} = y\vec{a_x} + (x+z)\vec{a_y}$ express P and $\vec{A}$ at point 'P' in the Cartesian & cylindrical & spherical systems.

Sol : consider co-ordinate system (cartesian)

$P(x,y,z) \Rightarrow x=-2 ; y=6 ; z=3$

$P(-2,6,3)$

cylindrical $\Rightarrow$ $\rho = \sqrt{x^2+y^2} = 6.32$

$\phi = \tan^{-1}(y/x) = 108.43°$

$z = 3$

spherical $\Rightarrow$ $r = \sqrt{x^2+y^2+z^2} = 7$

$\theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z} = 64.62°$

$\phi = 108.43°$

Vector $\vec{A}$ in cartesian co-ord system

$\vec{A} = 6\vec{a_x} + \vec{a_y}$

Cylindrical : substitute the values of $\vec{a_x}$ & $\vec{a_y}$ in terms of a_ρ & a_ϕ

$x = \rho\cos\phi$; $y = \rho\sin\phi$ $z = z$

$\cos\phi = \frac{-2}{\sqrt{40}}$; $\sin\phi = \frac{6}{\sqrt{40}}$

$\vec{A} = -0.9487\vec{a_\rho} - 6.008\vec{a_\phi}$

Spherical : $\vec{A} = -0.8571\vec{a_r} - 0.4066\vec{a_\theta} - 6.008\vec{a_\phi}$

Note : $|A|$ or magnitude of $\vec{A}$

Cartesian : $|A| = \sqrt{36+1} = 6.803$

Cylindrical : $|A| = 6.803$

Spherical : $|A| = 6.803$

$|A|$ is the same in the three systems i.e

$|A(x,y,z)| = |A(\rho,\phi,z)| = |A(r,\theta,\phi)| = 6.803$

SUMMARY OF Vectors & co-ordinate system :

1> dot product of same unit Vectors is equal to '1'

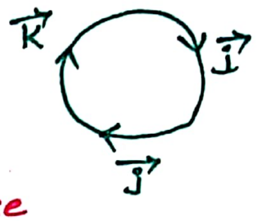
$$\vec{i}\cdot\vec{i} = \vec{j}\cdot\vec{j} = \vec{k}\cdot\vec{k} = 1 \quad \&$$

different unit Vectors

$$\vec{i}\cdot\vec{j} = \vec{j}\cdot\vec{k} = \vec{k}\cdot\vec{i} = 0$$

2> Scalar (on) dot product $\vec{A}\cdot\vec{B} = |A||B|\cos\theta$

3> Vector on cross product $\vec{A}\times\vec{B} = |A||B|\sin\theta$



$$\vec{i}\times\vec{i} = \vec{j}\times\vec{j} = \vec{k}\times\vec{k} = 0$$

$$\left.\begin{aligned} \vec{i}\times\vec{j} &= \vec{k} \\ \vec{j}\times\vec{k} &= \vec{i} \\ \vec{k}\times\vec{i} &= \vec{j} \end{aligned}\right\} \qquad \left.\begin{aligned} \vec{j}\times\vec{i} &= -\vec{k} \\ \vec{k}\times\vec{j} &= -\vec{i} \\ \vec{i}\times\vec{k} &= -\vec{j} \end{aligned}\right\} \text{Reverse rotation}$$

4> gradient $\nabla\phi = \left[\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right]\phi$

$$= \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

5> Divergence theorem [Surface ⟷ Volume] $\oint_S A\cdot ds = \int_V (\nabla\cdot A)\,dv$

$$\nabla\cdot A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$ for cartesian co-ordinate system

$$\nabla\cdot A = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

for cylindrical

$$\nabla\cdot A = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

for Spherical

6) Laplacian of a scalar

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \quad \text{for rectangular}$$

7) curl (or) rotation :

$$\nabla \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \quad \text{for rectangular}$$

$$\nabla \times \vec{A} = \frac{1}{\rho} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix} \quad \text{for cylindrical}$$

$$\nabla \times \vec{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$

for spherical

8) differential length

$$dl = dx\,\vec{a_x} + dy\,\vec{a_y} + dz\,\vec{a_z}$$

for rectangular

ds along $\vec{a_x}$ = $dy\,dz\,\vec{a_x}$

" " $\vec{a_y}$ = $dx\,dz\,\vec{a_y}$

" " $\vec{a_z}$ = $dx\,dy\,\vec{a_z}$

$d\vec{v} = dx\,dy\,dz$

lll'y

$$dl = d\rho\,\vec{a_\rho} + \rho\, d\phi\,\vec{a_\phi} + dz\,\vec{a_z}$$

ds along $\vec{a_\rho}$ = $\rho\, d\phi\, dz\,\vec{a_\rho}$

$\vec{a_\phi}$ = $d\rho\, dz\,\vec{a_\phi}$

$\vec{a_z}$ = $\rho\, d\rho\, d\phi\,\vec{a_z}$

$d\vec{v} = \rho\, d\phi\, d\rho\, dz$

III'y $\quad dl = dr\,\vec{a_r} + r\,d\theta\,\vec{a_\theta} + r\sin\theta\,d\phi\,\vec{a_\phi}$

ds along $\vec{a_r} = r^2\sin\theta\,d\theta\,d\phi\,\vec{a_r}$

" " $\vec{a_\theta} = r\sin\theta\,dr\,d\phi\,\vec{a_\theta}$

" " $\vec{a_\phi} = r\,dr\,d\theta\,\vec{a_\phi}$

$$\boxed{dv = r^2\sin\theta\,dr\,d\theta\,d\phi}$$

9> Stokes theorem $\rightarrow$ $\oint_L \vec{A}\cdot\vec{dl} = \iint_S (\nabla\times\vec{A})\cdot\vec{ds}$

Problems :

$*$

① Determine the Vector $\vec{A}$ directed from $(2,-4,1)$ to $(0,-2,0)$ in cartesian coordinates and also determine the unit vector $\vec{A}$

Sol :

A to B $\boxed{B-A}$

$$\vec{A} = (0-2)\vec{a_x} + (-2+4)\vec{a_y} + (0-1)\vec{a_z}$$

$$= -2\vec{a_x} + 2\vec{a_y} - \vec{a_z}$$

Unit Vector along $\vec{A} \rightarrow \vec{a_A} = \frac{\vec{A}}{|A|}$

$$= \frac{-2\vec{a_x} + 2\vec{a_y} - \vec{a_z}}{\sqrt{4+4+1}}$$

$$= -2/3\,\vec{a_x} + 2/3\,\vec{a_y} - 1/3\,\vec{a_z}$$

② use spherical Co-ordinate system to find area of the strip $\alpha \le \theta \le \beta$ on the shell of radius 'r'. what results when $\alpha = 0$ & $\beta = \pi$

Sol :

$$\vec{ds} \text{ along } \vec{a_r} = r^2 \sin\theta \, d\theta \, d\phi$$

$$= \int_{\theta=\alpha}^{\beta} \sin\theta \, d\theta \int_{\phi=0}^{2\pi} d\phi \times r^2$$

$$= r^2 \times 2\pi \times -[\cos\beta - \cos\alpha]$$

$$= 4\pi r^2 \; [\text{when } \alpha = 0 \;\&\; \beta = \pi]$$

③ use cylindrical Co-ordinate system to find the area of the curved surface of a cylinder where $\rho = 2m$, $z = 5m$ and ϕ varies from $30°$ to $120°$

Sol :

$$\vec{ds} \text{ along } \vec{a_\rho} = \rho \times \int_0^5 dz \times \int_{\phi=30°}^{120°} d\phi$$

$\rho \, d\phi \, dz$

$$= 5 \times 2 \times \frac{\pi}{2}$$

$$= 5\pi \; m^2$$

④ Find the Force on charge $Q_1 = 20\mu C$ due to charge $Q_2 = -300\mu C$ where Q_1 is at $(0,1,2)\,m$ and Q_2 is at $(2,0,0)\,m$

Sol :

$$\vec{F} = \frac{1}{4\pi \times \frac{1}{36\pi \times 10^{-9}}} \times \frac{20\times10^{-6} \times -300\times10^{-6}}{3} \times \frac{-2\vec{a_x} + \vec{a_y} + 2\vec{a_z}}{}$$

Q_1 due to Q_2

$\boxed{Q_1 - Q_2}$

$\vec{R} = -2\vec{a_x} + \vec{a_y} + 2\vec{a_z}$

$\vec{a_R} = \downarrow / 3$

$$= 4\vec{a_x} - 2\vec{a_y} - 4\vec{a_z} \text{ Newtons.}$$

Cartesian Co-ordinates

① Differential displacement

$$\vec{dl} = dx\,\vec{a_x} + dy\,\vec{a_y} + dz\,\vec{a_z}$$

② Differential normal area is given by
on Surface

$$\vec{ds} = dy\,dz\,\vec{a_x}$$
$$= dx\,dz\,\vec{a_y}$$
$$= dx\,dy\,\vec{a_z}$$

③ Differential volume is given by

$$\vec{dv} = dx\,dy\,dz$$

Cylindrical Co-ordinates :

① differential displacement is given by

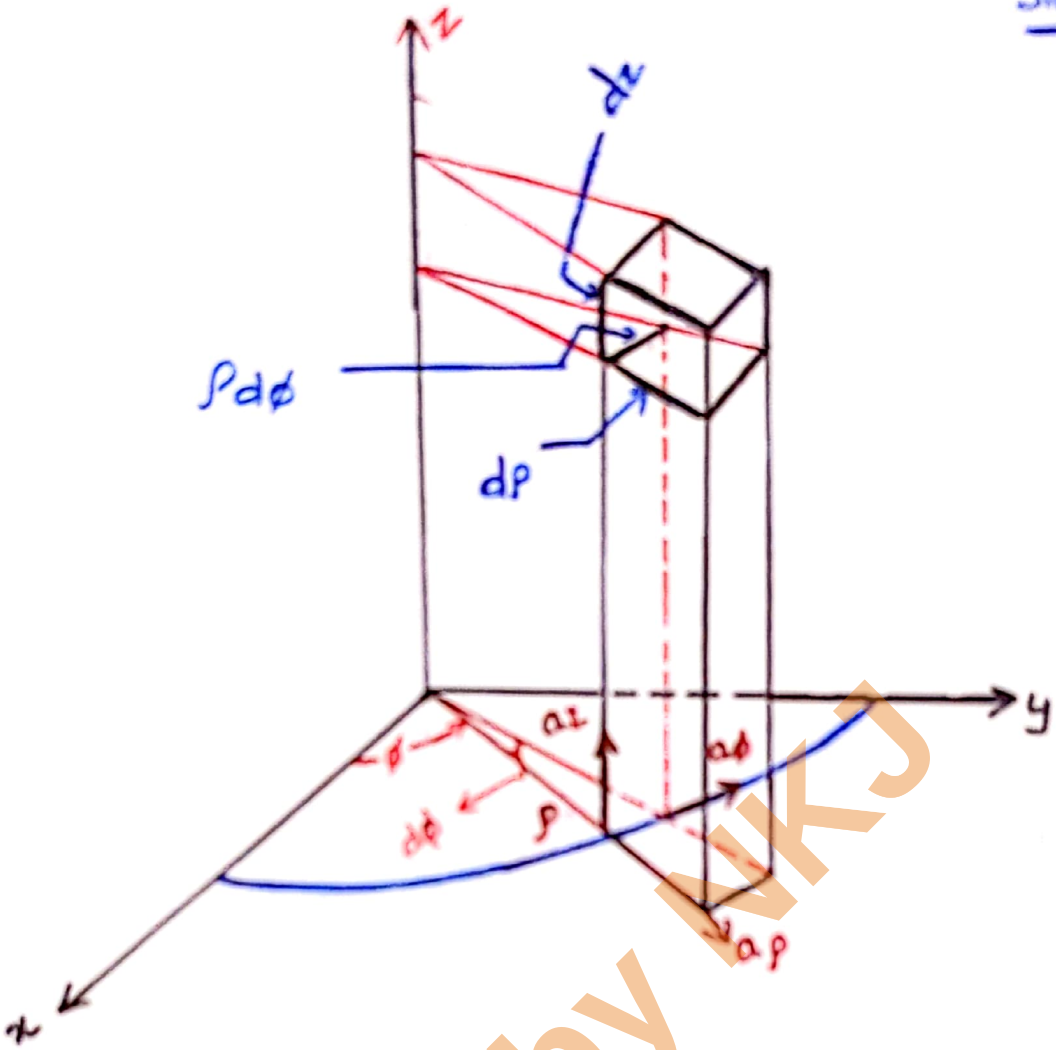
$$\vec{dl} = d\rho\,\vec{a_\rho} + \rho\,d\phi\,\vec{a_\phi} + dz\,\vec{a_z}$$

② differential normal area is given by

$$\vec{ds} = \rho\,d\phi\,dz\,\vec{a_\rho}$$
$$= d\rho\,dz\,\vec{a_\phi}$$
$$= \rho\,d\phi\,d\rho\,\vec{a_z}$$

③ Differential volume is given by

$$\vec{dv} = \rho\,d\phi\,d\rho\,dz$$

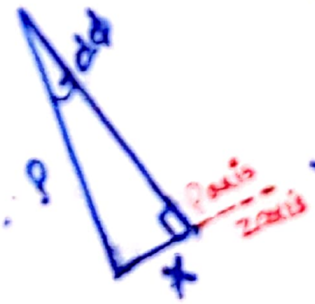


dl along a_z is $dz\,\vec{a_z}$

dl along a_ρ is $d\rho\,\vec{a_\rho}$

dl along a_ϕ is $\rho\, d\phi\, \vec{a_\phi}$

$\sin d\phi = \frac{x}{\rho}$



$x = \rho \sin d\phi$

$= \rho\, d\phi$

ρ & z ⊥lar to each other so 90°

$$\frac{\rho_L}{4\pi\varepsilon_0}\int \frac{(\rho\tan\alpha\,\vec{a_z} + \rho\vec{a_\rho})(-\rho\sec^2\alpha)\,d\alpha}{(\rho^2)^{3/2}(1+\tan^2\alpha)^{3/2}}$$

$$= \frac{\rho_L}{4\pi\varepsilon_0}\int \frac{\left(\rho\frac{\sin\alpha}{\cos\alpha}\vec{a_z} + \rho\vec{a_\rho}\right)(-\rho\sec^2\alpha)\,d\alpha}{(\rho^3)(\sec^3\alpha)}$$

$$= \frac{\rho_L}{4\pi\varepsilon_0}\int \frac{\rho[\cos\alpha\,\vec{a_\rho} + \sin\alpha\,\vec{a_z})(-\rho\sec^2\alpha)\,d\alpha}{(\rho^3)(\sec^3\alpha)\cos\alpha}$$

$$= -\frac{\rho_L}{4\pi\varepsilon_0\rho}\int_{\alpha_1}^{\alpha_2}[\cos\alpha\,\vec{a_\rho} + \sin\alpha\,\vec{a_z}]\,d\alpha$$

Thus for a finite line charge,

$$E = \frac{\rho_L}{4\pi\varepsilon_0\rho}\left[-(\sin\alpha_2 - \sin\alpha_1)\vec{a_\rho} + (\cos\alpha_2 - \cos\alpha_1)\vec{a_z}\right]$$

For an infinite line charge point B is at $(0,0,\infty)$ and A at $(0,0,-\infty)$ so that $\alpha_1 = \pi/2$, $\alpha_2 = -\pi/2$ & E becomes

***

$$\boxed{E = \frac{\rho_L}{2\pi\varepsilon_0\rho}\vec{a_\rho}}$$

SURFACE CHARGE:-

Consider an infinite sheet of charge in the x-y plane with uniform charge density ρ_s. The charge associated with an elemental area "ds" is

$$dQ = \rho_s\,ds \quad \&$$

$$\rho_s = Q/s$$

$$dE = \frac{dQ}{4\pi\varepsilon_0 R^2}\vec{a_R}$$

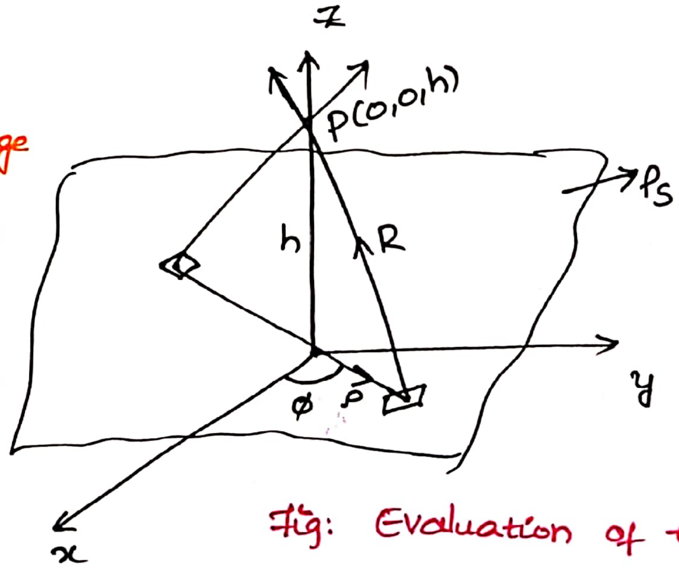
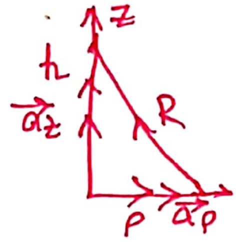


Fig: Evaluation of the E field due to an infinite sheet of charge.



$$\rho\vec{a_\rho} + \vec{R} = h\vec{a_z}$$

$$\vec{R} = \rho(-\vec{a_\rho}) + h\vec{a_z}$$

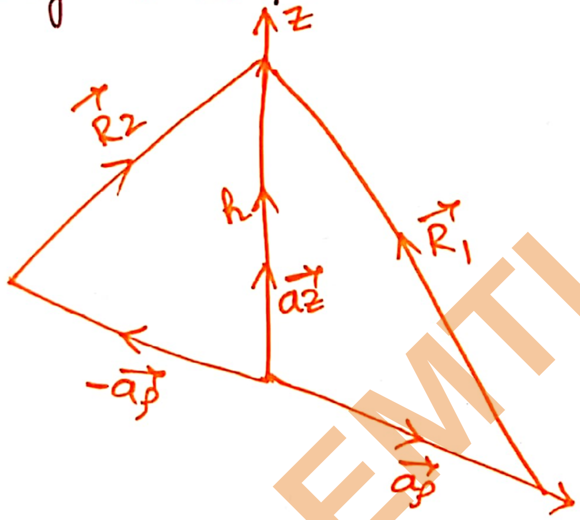
$$|R| = [\rho^2 + h^2]^{1/2}$$

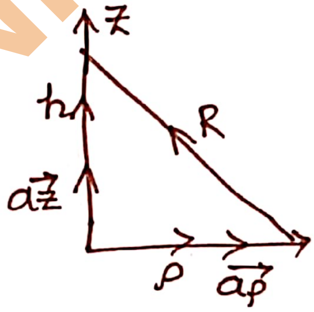
ds along z- direction = $\rho d\rho d\phi$

$$dQ = \rho_s ds = \rho_s \rho d\rho d\phi$$

$$dE = \frac{\rho_s \rho d\rho d\phi \, [-\rho\vec{a_\rho} + h\vec{a_z}]}{4\pi\varepsilon_0 [\rho^2 + h^2]^{3/2}}$$

Due to the symmetry of the charge distribution, for every element 1, there is a corresponding element 2, whose contribution along a_ρ cancels that of element 1 as illustrated in fig: Thus the contributions to E_ρ add up to zero so that E has only z-component.





$$\vec{R_1} = \rho(-\vec{a_\rho}) + h\vec{a_z}$$

$$\vec{R_2} = \rho(\vec{a_\rho}) + h\vec{a_z}$$

$$\vec{R_1} + \vec{R_2} = 2h\vec{a_z}$$

$$E = \int dE_z = \frac{\rho_s}{4\pi\varepsilon_0} \int_{\phi=0}^{2\pi} \int_{\rho=0}^{\infty} \frac{h\rho d\rho d\phi}{[\rho^2 + h^2]^{3/2}} \vec{a_z}$$

considering 2h the limits are $\int_0^{\pi}$

h the limits are $\int_0^{2\pi}$

$$= \frac{\rho_s h}{4\pi\varepsilon_0} 2\pi \int_0^{\infty} [\rho^2 + h^2]^{-3/2} \frac{1}{2} d(\rho^2) \vec{a_z}$$

$$= \frac{\rho_s h}{2\varepsilon_0} \left\{ -[\rho^2 + h^2]^{-1/2} \right\}_0^{\infty} \vec{a_z}$$

$$E = \frac{\rho_s}{2\varepsilon_0} \vec{a_z}$$

In general for an infinite sheet of charge

$$\boxed{E = \frac{\rho_s}{2\varepsilon_0} \vec{a_n}}$$

unit vector $\perp$ an to the sheet

INTRODUCTION:

* Electrostatic fields are also called static electric fields (or) Steady electric fields.
* These fields are not variant with time.
* They are produced by static charges (or) charge distributions

Applications :-

Electrostatic fields are used

→ in CRO's to obtain the electron beam deflection.

→ in ink-jet printers to obtain speed of printing and quality of print.

→ to produce potential

→ to produce force on charges for their mobility.

→ in FET's and Capacitors

→ in LCD's and touching pad's

→ in medical applications like ECGs & X-ray machines.

→ in Computer pheripheral's.

Different types of Charge distributions :-

* charges at rest produce electrostatic field. The charges are basically of two types : positive & negative.
* charges are moving with constant velocity produce magnetic field.
* When a charge is accelerated, electromagnetic field is produced. A fraction of the field is detached and propagates at the speed of Light. The detached field carries energy, momentum & angular momentum. This is called electromagnetic radiation.
* charge is conserved. It can neither be created nor destroyed.
* charges are surrounded by electric and magnetic fields
* A charge experiences a force in the presence of a field.

(i) Point charges, Q (coulomb) :-

These are the charges which do not occupy any space, i.e the Volume of the point charge is zero. For ex: an electron is considered to be a point charge and has a charge of 1.6×10^{-19} C.

(ii) Line charge distribution, ρ_L (c/m) :-

This is a charge distribution in which the charge is distributed along a line like a filament, i.e this has only Length but no Width or breadth. ρ_L is the (defined) as the charge per unit Length.

$$\rho_L = \frac{dQ}{dL}$$

Ex : electron beam in CRT.

(iii) Surface charge distribution (ρ_s) (c/m²) :-

When a charge is confined to the surface of a conductor, it is said to be surface charge distribution. Such a surface has both Length and width but no breadth.

Surface density is defined as the charge per unit area i.e

$$\rho_s = \frac{dQ}{ds} = \lim_{\Delta s \to 0} \frac{\Delta Q}{\Delta s}$$

Ex : is the conductor surface of a capacitor.

(iv) Volume charge distribution, ρ_v (c/m³) :-

Volume charge density is defined as the charge per unit Volume, i.e

$$\rho_v = \lim_{\Delta v \to 0} \frac{\Delta Q}{\Delta v} = \frac{dQ}{dv}$$

Ex :- ionospheric region, electron cloud in vaccum tube.

UNIT-1

Electrostatics :-

Coulomb's Law :-

→ is an Experimental Law formulated in 1785 by the French colonel charles Augustin d.c coulomb.

Coulomb's Law states that the force 'F' between two point charges Q_1 & Q_2 is :

1> Along the line joining them
2> directly proportional to the product of the charges $Q_1 Q_2$
3> Inversely proportional to the square of the distance 'R' between them

Expressed Mathematically

$$F = \frac{KQ_1Q_2}{R^2} \quad \text{--- (1)}$$

Where 'K' is the proportionality constant and

$$K = \frac{1}{4\pi\varepsilon} = \frac{1}{4\pi\varepsilon_0\varepsilon_r}$$

Where ε_0 is the permitivity of free space (farads/m)
ε_r " " relative " " the medium w.r. to freespace
$\varepsilon_r = 1$ for air (or) free space.

In S.I units (system internations)
The units of charges Q_1 & Q_2 are coulombs (C)
" " " distance R in meters (m)
" " " Force F in Newtons (N)

The value of $\varepsilon_0 = 8.854 \times 10^{-12} \approx \frac{10^{-9}}{36\pi}$ F/m

then $K = 10^9 \times 9$ m/F

& 1 coulomb is approximately equal to 6×10^{18} electrons

Eqn (1) becomes $F = \frac{Q_1Q_2}{4\pi\varepsilon_0 R^2}$ --- (2)

Electrostatics :-

coulomb's Law :

→ is an experimental Law formulated in 1785 by the French colonel, charles Augustin de coulomb.

Coulomb's law states that the Force 'F' between two Point charges Q_1 and Q_2 is :

1> Along the line joining them

2> directly Proportional to the product $Q_1 Q_2$ of the charges

3> Inversely Proportional to the square of the distance 'r' between them.

Expressed mathematically $\boxed{F = \frac{K Q_1 Q_2}{r^2}}$ —— (1)

Where 'K' is the proportionality constant and

$$K = \frac{1}{4\pi\varepsilon_0\varepsilon_r} = \frac{1}{4\pi\varepsilon_0} = 10^9 \times 9 \ m/F$$

Where ε_0 is the permitivity of free space (farads/m)

ε_r is the relative permitity of the medium w.r.to free space

$\varepsilon_r = 1$ for air on free space

In S.I units (system internations)

> The units of charges Q_1 & Q_2 are coulombs (C)

> " " " distance r in meters (m)

> " " " Force F in Newtons (N)

The value of $\varepsilon_0 = 8.854 \times 10^{-12} = 10^{-9}/36\pi$ F/m and 1 coulomb is approximately equal to 6×10^{8} electrons.

Since Force is a vector quantity having magnitude and direction both and eqn (1) can be written as

$$\boxed{\vec{F} = \frac{Q_1 Q_2}{4\pi\varepsilon r^2}\vec{a_r}} \quad (\text{or}) \quad \boxed{\vec{F} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 r^2}\vec{a_r}} \text{ when } \varepsilon_r = 1$$

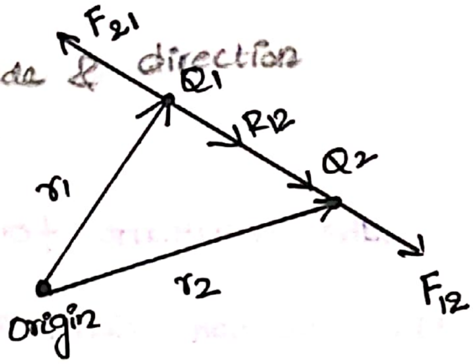
Where $\vec{a_r}$ is the unit vector in the direction of line joining of charge.

Eqn (1) becomes $F = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2}$ —— (2)

COULOMB's LAW .... contd...

Since Force is a vector quantity having magnitude & direction

→ If Point charges Q_1 and Q_2 are located at Points having position vectors r_1 and r_2, then the Force F_{12} on Q_2 due to Q_1 shown in fig: is given by



$$F_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2}\, a_{R12} \qquad \text{where } R_{12} = r_2 - r_1$$

$$R = |R_{12}|$$

$$\therefore \quad F_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^3} R_{12} \quad \left(\text{replace } a_{R12} = \frac{R_{12}}{R}\right) \qquad \boxed{a_{R12} = \frac{R_{12}}{R}}$$

$$F_{12} = \frac{Q_1 Q_2 (r_2 - r_1)}{4\pi\varepsilon_0 |r_2 - r_1|^3} \quad \left(\text{substitute } R_{12} = r_2 - r_1\right)$$

|||ly the Force F_{21} on Q_1 due to Q_2 is given by

$$F_{21} = -F_{12}$$

Limitations :-

→ The distance between two charged bodies Q_1 & Q_2 must be Large compared to Linear dimensions of the charges on bodies i.e Q_1 & Q_2 must be Point charges.

→ applicable only when two charges are at Rest.

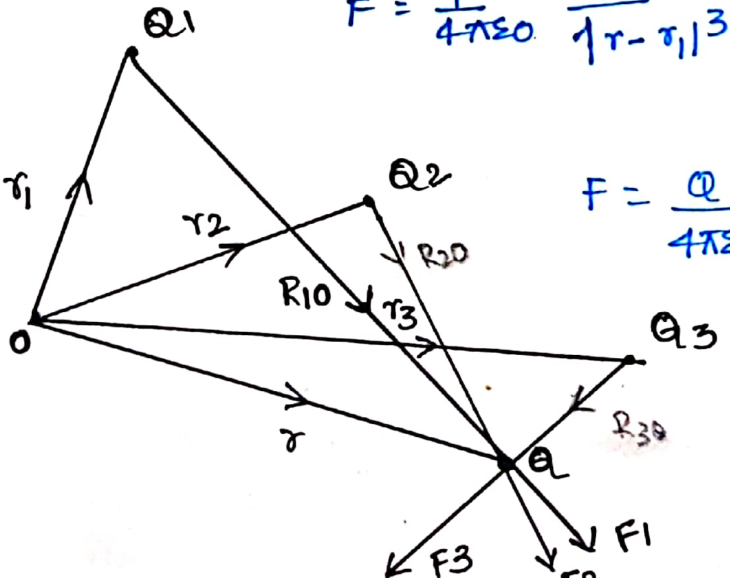
Force due to 'N' number of charges :-

$$F = \frac{1}{4\pi\varepsilon_0} \frac{Q_1 Q}{|r - r_1|^3}(r - r_1) + \frac{1}{4\pi\varepsilon_0} \frac{Q_2 Q}{|r - r_2|^3}(r - r_2) + \dots$$

$$-r + r_1 + R_{10} = 0$$

$$R_{10} = r - r_1$$

$$F = F_1 + F_2 + F_3$$



$$F = \frac{Q}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{Q_i}{|r - r_i|^3}(r - r_i)$$

Electric field intensity (or Electric field strength (E)):-

→ If a small test charge (or probe charge 'q' is placed at any point near the second fixed charge 'Q'. The probe charge 'q' experiences a Force (Between two charged particles force exists).

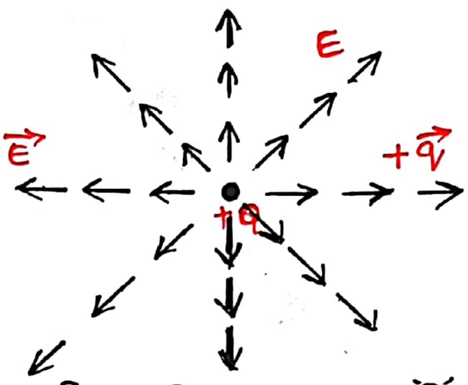


fig: fixed charge 'Q' With vectors showing magnitude & direction of associated electric field.

→ The magnitude & direction of this force depends upon the Location of the probe charge (q) w.r. to fixed charge (Q).

→ About the charged 'Q' an electric field of Strength 'E' exists.

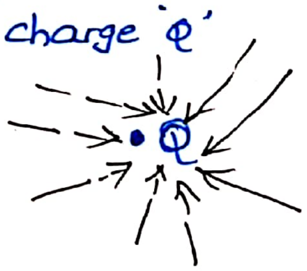
→ The magnitude of E at any point is measured as Force per unit charge at that Point.

$$\boxed{E = \frac{F}{Q}}$$ → Vector / → Vector

∴ $\vec{E}$ is a Vector quantity.

$\vec{E}$ is the Force per unit charge when placed in the electricfield

Before placing charge Q the charged particles around the charge are constant. After placing a fixed charge 'Q' near the charged particles, the charged particles are influenced by the fixed charge and the field lines are created around the fixed charge 'Q'



$$\vec{E} = \frac{\cancel{Q_1} Q_2}{4\pi\varepsilon r^2} \times \frac{1}{\cancel{Q}} \vec{a_r}$$

$\boxed{Q_1 = Q_2 = Q}$

$$\boxed{\vec{E} = \frac{Q}{4\pi\varepsilon r^2} \vec{a_r}}$$

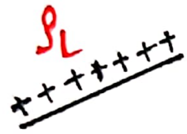
Note : $\vec{E}$ is independent of the probe charge 'q' because of probe charge 'q' Electric field lines are Created around a fixed charge. ↓ ↓ ↙ around the probe charge (q) is zero.

'E' is the force per unit charge when placed in the electric field.

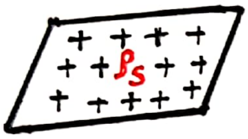
→ If the charge on the test charge is allowed to approach zero then the electric field intensity remains constant because $\vec{E}$ is independent of q.

Electric fields due to continuous charge distributions :-

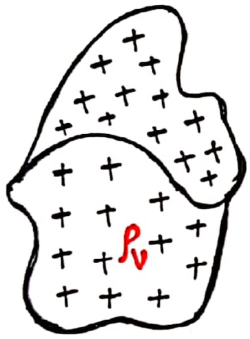
→ Point charge — coulomb (C) — Q — $\vec{E} = \dfrac{Q}{4\pi\varepsilon_0 r^2}\vec{a_r}$



→ Line charge ρ_L — C/m — $dQ = \rho_L dl$ — $Q = \int_L \rho_L dL$ — $\vec{E} = \int \dfrac{\rho_L dL}{4\pi\varepsilon_0 r^2}\vec{a_r}$



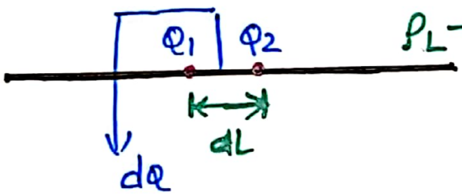
→ Surface charge ρ_S — C/m² — $dQ = \rho_S ds$ — $Q = \int_S \rho_S ds$ — $\vec{E} = \int \dfrac{\rho_S ds}{4\pi\varepsilon_0 r^2}\vec{a_r}$



→ Volume charge ρ_V — C/m³ — $dQ = \rho_V dv$ — $Q = \int_V \rho_V dv$

$$\vec{E} = \int \frac{\rho_V dv}{4\pi\varepsilon_0 r^2}\vec{a_r}$$

Consider a line with uniform charge density ρ_L C/m. The line having two point charges Q_1 & Q_2. the distance b/wn two point charges is dL and the charge element dQ associated with the element 'dL' of the line is



ρ_L → total line charge.

$$dQ = \rho_L \cdot dL$$

IIIly The charge dQ associated with an elemental area ds is $dQ = \rho_S ds$ &

The charge dQ associated with an elemental volume dv is $dQ = \rho_V dv$

A. A Line charge:

consider a Line charge with uniform charge density 'ρ_L' extending from $(0,0,z')$ A to B along the z-axis.

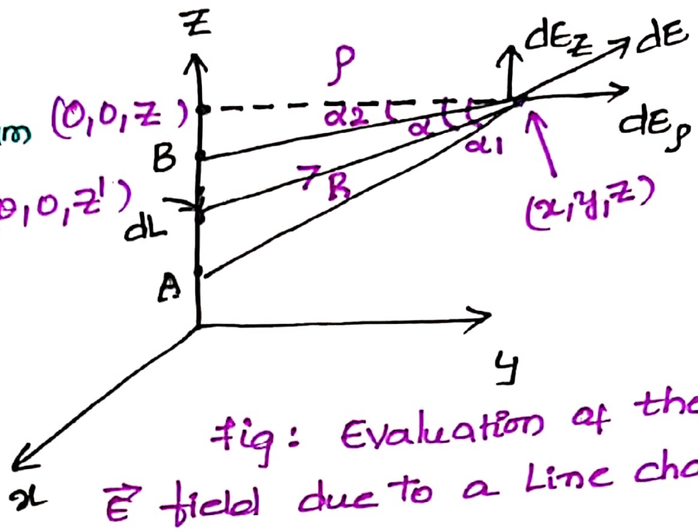


fig: Evaluation of the $\vec{E}$ field due to a Line charge

$$\rho_L = \frac{dQ}{dL} \Rightarrow dQ = \rho_L dL$$

$$Q = \int \rho_L dL$$

$$= \int_{z_A}^{z_B} \rho_L dz'$$

$$R = (x, y, z) - (0,0,z')$$

$$= x\vec{a_x} + y\vec{a_y} + (z-z')\vec{a_z}$$

convert into cylindrical

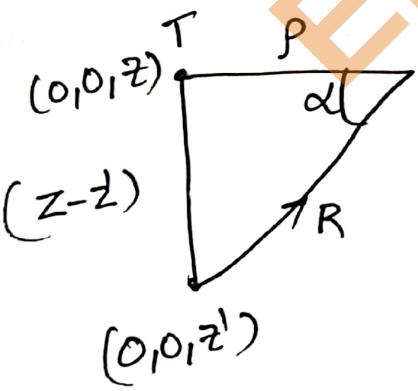
$$x = \rho\cos\phi \ , \ y = \rho\sin\phi \qquad \vec{a_x} = \cos\phi\,\vec{a_\rho} - \sin\phi\,\vec{a_\phi}$$

$$\vec{a_y} = \sin\phi\,\vec{a_\rho} + \cos\phi\,\vec{a_\phi}$$

After simplifying $\quad R = \rho\vec{a_\rho} + (z-z')\vec{a_z}$

$$|R| = \left[\rho^2 + (z-z')^2\right]^{1/2}$$

$$E = \int \frac{\rho_L dl}{4\pi\varepsilon_0 R^2}\vec{a_R} = \frac{\rho_L}{4\pi\varepsilon_0}\int \frac{\rho\vec{a_\rho} + (z-z')\vec{a_z}}{\left[\rho^2 + (z-z')^2\right]^{3/2}} dz'$$



$$\tan\alpha = \frac{z-z'}{\rho} \Rightarrow z-z' = \rho\tan\alpha$$

$$\cos\alpha = \rho/R$$

$$R = \rho\sec\alpha$$

$$E = \frac{\rho_L}{4\pi\varepsilon_0}\int \frac{\rho(\vec{a_\rho} + \tan\alpha\,\vec{a_z})(-\rho\sec^2\alpha)\,d\alpha}{(\rho^2 + \rho^2\tan^2\alpha)^{3/2}}$$

Force due to several point charges :-

→ If we have more than two point charges, superposition theorem is used to determine the force on a particular charge. The principle states that if there are 'N' charges $Q_1, Q_2, \ldots, Q_N$ located respectively at points with position vectors $r_1, r_2, \ldots, r_N$, the resultant force 'F' on a charge 'Q' located at point 'r' is the vector sum of forces exerted on 'Q' by each of the charges $Q_1, Q_2, \ldots Q_N$ hence

II.

$$\vec{F} = \frac{Q Q_1 (r - r_1)}{4\pi\varepsilon_0 |r - r_1|^3} + \frac{Q Q_2 (r - r_2)}{4\pi\varepsilon_0 |r - r_2|^3} + \cdots + \frac{Q Q_N (r - r_N)}{4\pi\varepsilon_0 |r - r_N|^3}$$

$$\vec{F} = \frac{Q}{4\pi\varepsilon_0} \sum_{K=1}^{N} \frac{Q_K (r - r_K)}{|r - r_K|^3}$$

III|4

$$\vec{E} = \frac{\vec{F}}{Q} = \frac{1}{4\pi\varepsilon_0} \sum_{K=1}^{N} \frac{Q_K (r - r_K)}{|r - r_K|^3}$$

Problems :-

① Point charges 1 mC and −2 mC are located at (3, 2, −1) and (−1, −1, 4) respectively. Calculate the electric force on a +10 nC charge located at (0, 3, 1) and the electric field intensity at that point.

Sol: $Q = 10\times10^{-9}\,C$; $Q_1 = 1\times10^{-3}\,C$; $Q_2 = -2\times10^{-3}\,C$

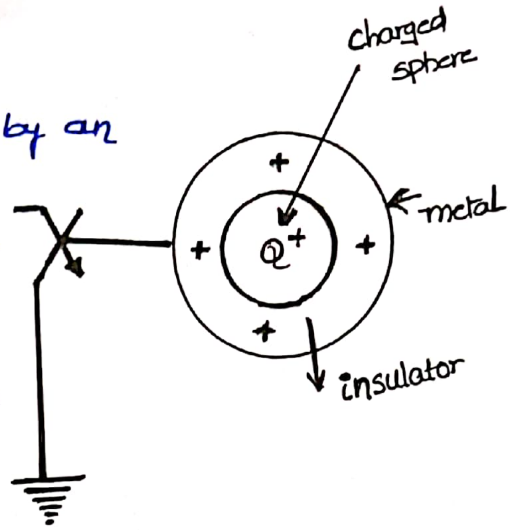
$r = (0, 3, 1)$; $r_1 = (3, 2, -1)$; $r_2 = (-1, -1, 4)$

$$\vec{F} = \frac{10\times10^{-9}\times1\times10^{-3}}{4\pi\times\frac{1}{36\pi\times10^{-9}}} \frac{(-3\vec{a_x} + \vec{a_y} + 2\vec{a_z})}{14\sqrt{14}} + \frac{10\times10^{-9}\times-2\times10^{-3}}{4\pi\times\frac{1}{36\pi\times10^{-9}}} \times \frac{(\vec{a_x} + 4\vec{a_y} - 3\vec{a_z})}{26\sqrt{26}}$$

$$\vec{F} = (-6.507\,\vec{a_x} - 3.817\,\vec{a_y} + 7.506\,\vec{a_z}) \times 10^{-3} \text{ Newtons}$$

$$\vec{E} = \vec{F}/Q = (-650.7\,\vec{a_x} - 381.7\,\vec{a_y} + 750.6\,\vec{a_z})\ \text{kV/m}$$

Faraday's experiment :-



→ Two concentric conducting spheres are seperated by an insulating material. (air, glass (or) wood)

→ The inner sphere is charged to $\boxed{+Q}$. The outer sphere is initially uncharged.

→ The outer sphere is grounded momentarily.

→ The charge on the outer surface is found to be $\boxed{-Q}$.

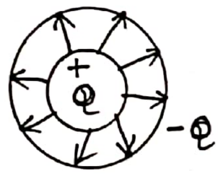
that means charge was found to be equal and opposite sign to the charge on inner sphere.

→ Faraday concluded there was a displacement (or) electric displacement from the charge on the inner sphere through the inner sphere through the insulator to the outer surface and that displacement is equal to $-Q$.

→ The electric displacement (or electric flux) is equal in magnitude to the charge that produces it.

Electric displacement (or) Electric flux (ψ) :-

This is a quantity independent of the nature of the medium. According to Faraday's experiment



$$\boxed{\psi = Q} \text{ coulombs}$$

Electric flux density ($\vec{D}$) :-

W.K.T $\vec{F} = \dfrac{Q_1 Q_2}{4\pi\varepsilon_0 r^2}\vec{a_r}$ — ①

$\vec{E} = \dfrac{Q_1 Q_2}{4\pi\varepsilon_0 r^2} \times \dfrac{1}{Q}\vec{a_r}$

$= \dfrac{Q}{4\pi\varepsilon_0 r^2}\vec{a_r}$ — ②

→ from eqns ① & ② both

$\vec{F}$ & $\vec{E}$ $\propto \dfrac{1}{\varepsilon_0}$ → both dependent on the medium

→ Suppose ~~assume~~ Let us Define a new vector field $\vec{D}$ that it is independent of the medium is defined by

$$\boxed{\vec{D} = \varepsilon_0 \vec{E}}$$

$$= \cancel{\varepsilon_0}\frac{Q}{4\pi\cancel{\varepsilon_0} r^2} \Rightarrow \boxed{\vec{D} = \frac{Q}{4\pi r^2}} \ \text{C/m}^2$$

$\vec{D}$ is independent of the medium

differential length :

For ex:- We move from point 'P' to 'Q' (or Q to P), differential length $dl : dy\vec{a_y}$. because we are moving in the y-direction.

& if we move from 'Q' to 'S' (or 'S' to 'Q'),

$dl = dz\vec{a_z} + dy\vec{a_y}$

because first we are moving from Q to R along z-direction and from R to S along y-direction.

III ^{ly} We move from D to Q ***$\boxed{dl = dx\vec{a_x} + dy\vec{a_y} + dz\vec{a_z}}$

differential surface (or area) element ds may generally be defined as

$$dS = ds\,\vec{a_n}$$

Where ds is the area of the surface element and $\vec{a_n}$ is a unit vector normal to the surface ds.

consider surface ABCD in fig: $ds = dydz\,\vec{a_x}$

" " PQRS " . $ds = -dydz\,\vec{a_x}$

because $\vec{a_n} = -\vec{a_x}$ is normal to PQRS.

differential volume "dv" can be obtained from "dl" as the product of the three components of "dl" i.e dxdydz

***

Note : dl and ds are vectors where as dv is a scalar.

Once dL is remembered, ds & dv can easily be found.

For ex:- ds along a_x can be obtained from dL eqn by multiplying the components of dL along a_y & a_z.

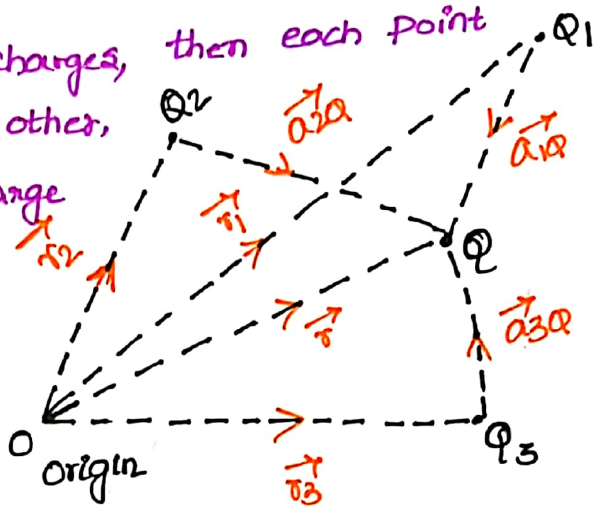
i.e $dy\,dy\,a_x$

Also <u>dv</u> can be obtained from dL as the product of the three components of <u>dL</u>

i.e $dx\,dy\,dz$.

Force due to several point charges:-

If there are morethan two Point charges, then each point charge will exert force on the other, then the net force on any charge can be obtained by using Superposition theorem.



Consider a point charge 'Q' surrounded by three other point charges Q_1, Q_2 & Q_3 as shown in fig.

→ The total force on $\vec{Q}$ is equal to the vector sum of all the forces exterted on 'Q' due to three point charges. Q_1, Q_2 & Q_3

First consider force exerted on Q due to Q_1. According to principle of superposition Q_2 & Q_3 are to be suppressed.

$$F_{Q_1Q} = \frac{Q_1 Q}{4\pi\varepsilon_0 R^2}\, a_{1Q} = \frac{Q_1 Q}{4\pi\varepsilon_0} \frac{(r-r_1)}{|r-r_1|^3}$$

$$F_{Q_2Q} = \frac{Q_2 Q}{4\pi\varepsilon_0} \frac{(r-r_2)}{|r-r_2|^3}$$

$$F_{Q_3Q} = \frac{Q_3 Q}{4\pi\varepsilon_0} \frac{(r-r_3)}{|r-r_3|^3}$$

$$F = F_{Q_1Q} + F_{Q_2Q} + F_{Q_3Q}$$

$$F = \frac{Q}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{Q_i (r-r_i)}{|r-r_i|^3}$$

Note : This Law is applicable only when two charges are at rest.

$$F_2 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2}\,\vec{a}_{R12}$$

$\vec{a}_{R12}$ = unit vector along $\vec{R_{12}}$ = $\frac{\text{Vector}}{\text{magnitude of Vector}}$

$$= \frac{\vec{R_{12}}}{|R_{12}|} = \frac{r_2 - r_1}{|r_2 - r_1|}$$

F_1 = Force exerted on Q_1 due to Q_2

$$F_1 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2}\,\vec{a}_{R21}$$

$R_{22} = R_{21} = \frac{r_1 - r_2}{|r_1 - r_2|}$

$\vec{a_{21}} = -\vec{a_{12}}$ (or)

$\vec{F_1} = -\vec{F_2}$

Hence force exerted by the two charges on each other is equal but opposite direction.

Note : 1. It is necessary that the two charges are the point charges & stationary in nature.

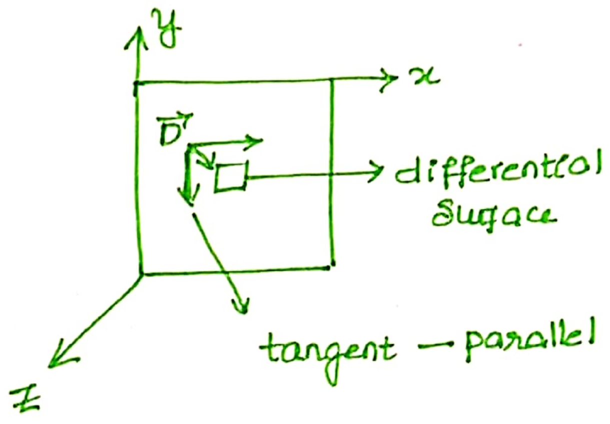
2. The two point charges may be +ve or −ve, hence their signs must be considered to calculate the force exerted.

tangent

$\vec{D} = D_x \vec{a_x}$ (or)

$\vec{D} = D_y \vec{a_y}$ (or)

$\vec{D} = D_x \vec{a_x} + D_y \vec{a_y}$



ds along $\vec{a_z}$ = $dx\,dy\,\vec{a_z}$

(Note : If the surface is considered along x & y direction it moves in the other direction i.e (remaining) $\vec{a_z}$

normal to the Surface

(or) ⊥ or " " "

$\vec{D} = D_z \vec{a_z}$

Now ⇒ $\vec{D} \cdot \vec{ds} = D_x \vec{a_x} \cdot dx\,dy\,\vec{a_z} = 0$ for tangential

⇒ $\vec{D} \cdot \vec{ds} = D_z \vec{a_z} \cdot dx\,dy\,\vec{a_z} = D_z\,dx\,dy$

$D\,ds = |D||ds|\cos\theta$

for normal

A Gaussian Surface is a closed surface in three dimensional space through which the flux of a vector field is calculated

the vector $\vec{D}$

Electric flux density

We define electric flux ψ in terms of D is

$$\psi = \int D \cdot ds$$

D is the Electric flux density

— independent of the medium

defined by $\vec{D} = \varepsilon_0 \vec{E}$

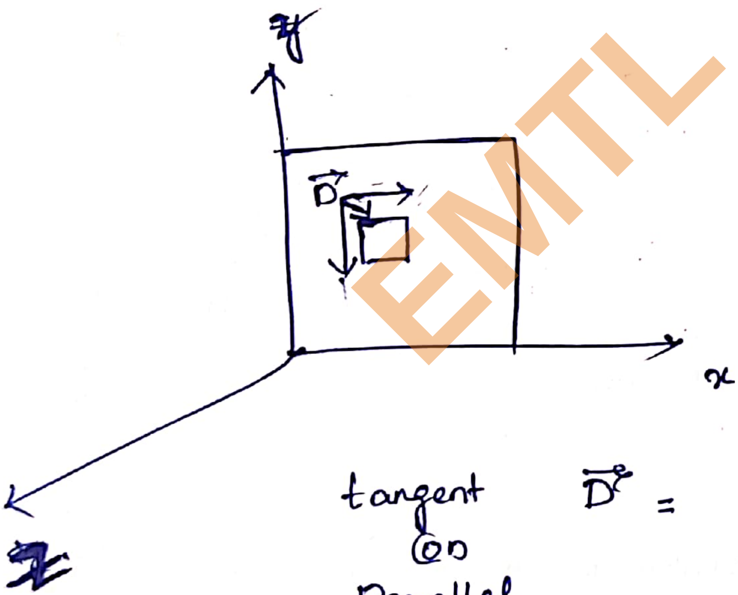
$$= \cancel{\varepsilon_0} \frac{Q}{4\pi \cancel{\varepsilon_0} r^2} = \frac{Q}{4\pi r^2}$$

$\varepsilon_0 \rightarrow$ Permittivity of the medium.

$$\rho_v = dq/dv$$

$$dq = \rho_v dv$$

$$Q = \int_v \rho_v dv.$$



Surface is considered along the x-y plane.

tangent (or) Parallel $\vec{D} = D_x a_x + D_y a_y$.

normal $\vec{D} = D_z a_z$

ds along $\vec{a_z}$ = $dx\, dy\, a_z$

$\vec{D} \cdot \vec{ds} = (D_x a_x + D_y a_y) \cdot dx\, dy\, a_z = 0$

$\vec{D} \cdot \vec{ds} = (D_z a_z)\, dx\, dy\, a_z = D_z\, dx\, dy.$

$|D||ds| \cos\theta = |D||ds|$

→ The surface integral is also called as the flux of vector $\vec{A}$

$$\phi = \psi = \iint_S \vec{A} \cdot \vec{ds} \quad \text{— (1)}$$

Substitute in eqn (1) vector $\vec{D}$ in place of vector $\vec{A}$ we get

$$\boxed{\psi = \iint_S \vec{D} \cdot \vec{ds}} \quad \text{— (2)}$$

In S.I units one line of electric flux entering from +1C and terminates on −1C. Therefore the electric flux is measured in coulombs. $\boxed{\psi = Q}$

Note: Some times Electric flux density $\vec{D}$ is called as electric displacement density (or) electric displacement.

Gauss Law :-

Statement 1: It states that the total displacement (or) electric flux through any closed surface surrounding the charge is equal to the amount of charge enclosed.

Symbolically $\boxed{\psi = Q}$

Proof: Let a positive charge 'Q' be placed at the center of an imaginary sphere of radius 'r'. The infinitesimal (small) amount of flux ($d\psi$) through the surface element ($\vec{ds}$) is

$$d\psi = \vec{D} \cdot \vec{ds}$$

If this integrated over the sphere of radius 'r' then the total electric flux (ψ) through the sphere is obtained i.e

$$\int d\psi = \iint \vec{D} \cdot \vec{ds} \quad \text{(or) from eqn (2)} \quad \boxed{\psi = \iint_S \vec{D} \cdot \vec{ds}}$$

$\vec{D}$ at any point is a function of charge 'Q' and position 'r' only.

$$\psi = \iint_S \frac{Q}{4\pi r^2} \cdot \vec{ds} = \iint \frac{Q}{4\pi r^2} \times r^2 \sin\theta \, d\theta \, d\phi$$

$\vec{ds}$ along $\vec{a_r}$ of the sphere $= r^2 \sin\theta \, d\theta \, d\phi$

$$= \frac{Q}{4\pi} \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \Rightarrow \boxed{\psi = Q_{enclosed}}$$

Gauss's Law :- States that

The total electric flux "ψ" through any closed surface is equal to the total charge enclosed by that surface.

$$\boxed{\psi = Q_{enc}}$$

For ex :-

from fig 1: The total flux Leaving v_1 is $10-5 = 5nC$ because only 10nC and $-5nC$ charges are enclosed by v_1

$$\boxed{\psi = 5nC}$$

from fig 2: The total flux Leaving v_2 is zero. because no charge is enclosed by v_2 $\boxed{\psi = 0}$

Note : "ψ" is independent of the charges outside the closed surface $\psi = \oint_S D \cdot ds = Q_{enc} = \int_v \rho_v \, dv$

By using Divergence theorem : [Surface $\longleftrightarrow$ Volume]

$$\oint_S A \cdot ds = \int_v (\nabla \cdot A) \, dv$$

$$\oint_S D \cdot ds = \int_v (\nabla \cdot D) \, dv$$

$$\int_v (\nabla \cdot D) \, dv = \int_v \rho_v \, dv \rightarrow \boxed{\rho_v = \nabla \cdot D}$$

Maxwell's 1^{st} eqn

States that the Volume charge density is the same as the divergence of the electric flux density.

Applications of Gauss's Law

→ once symmetric charge distributions exists. We construct a mathematical closed path is called a Gaussian Surface.

→ In the derivation of Gauss Law usually special Gaussian surfaces are used which has the following conditions.

1> The surface is closed

2> At each point of the surface $\vec{D}$ is either tangential or normal to the surface.

3> $\vec{D}$ has the same value at all points of the surface

4> When $\vec{D}$ is normal to the surface $\vec{D}\cdot\vec{ds} = |D||ds|$ → $\theta = 0°$

5> When $\vec{D}$ is tangertential to the surface $\vec{D}\cdot\vec{ds} = 0$ $\theta = 90°$

For this see backside of ↑ this page

These are the conditions to select a Gaussian surface.

1. Point charge :- Suppose a point charge 'Q' is located at the origin. To determine $\vec{D}$ at a point 'P', it is easy to see that choose a spherical surface containing 'P'.

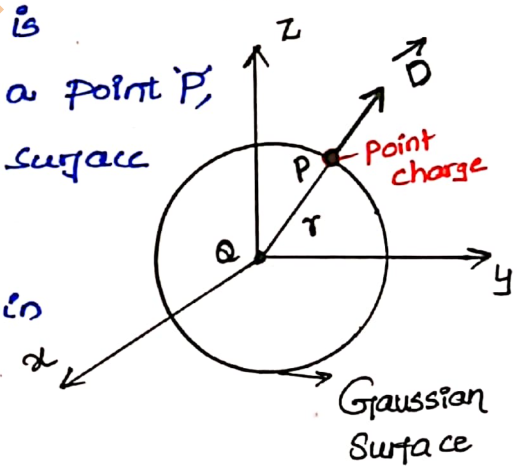


fig : Gaussian Surface about a point charge

→ A spherical surface centered at the origin is the Gaussian surface in this case.

→ As $\vec{D}$ is normal to the Gaussian surface everywhere

→ on the vector $\vec{D}$ is in r-direction and here we are using spherical co-ordinate system.

→ In spherical co-ordinate system the vector $\vec{D} = D_r\vec{a_r} + D_\theta\vec{a_\theta} + D_\phi\vec{a_\phi}$

According to the fig the vector $\vec{D}$ has no component or zero component along θ & ϕ direction. So $D_\theta\vec{a_\theta}$ and $D_\phi\vec{a_\phi}$ is equal to zero.

$$\boxed{\vec{D} = D_r\vec{a_r}}$$

$\vec{ds}$ along $\vec{a_r}$ is $r^2\sin\theta\, d\theta\, d\phi$

Applying Gauss Law $\oint_S \vec{D}\cdot\vec{ds} = Q_{enc}$

$$\iint D_r\vec{a_r}\cdot r^2\sin\theta\, d\theta\, d\phi\cdot\vec{a_r}$$

$$= r^2 D_r \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \quad [\vec{a_r}\cdot\vec{a_r} = 1]$$

$$Q = 4\pi r^2 D_r \Rightarrow D_r = \frac{Q}{4\pi r^2}$$

$$\boxed{\vec{D} = \frac{Q}{4\pi r^2}\vec{a_r}}.$$

II. Infinite Line charge :-

Suppose the infinite line of uniform charge ρ_L c/m lies along the z-axis.

→ To determine $\vec{D}$ at a point P, choose a cylindrical surface containing P.

→ $\vec{D}$ is normal to the surface anywhere.

$$\boxed{D_\rho \vec{a_\rho} = \vec{D}}$$

($\vec{D}$ is in ρ direction only)

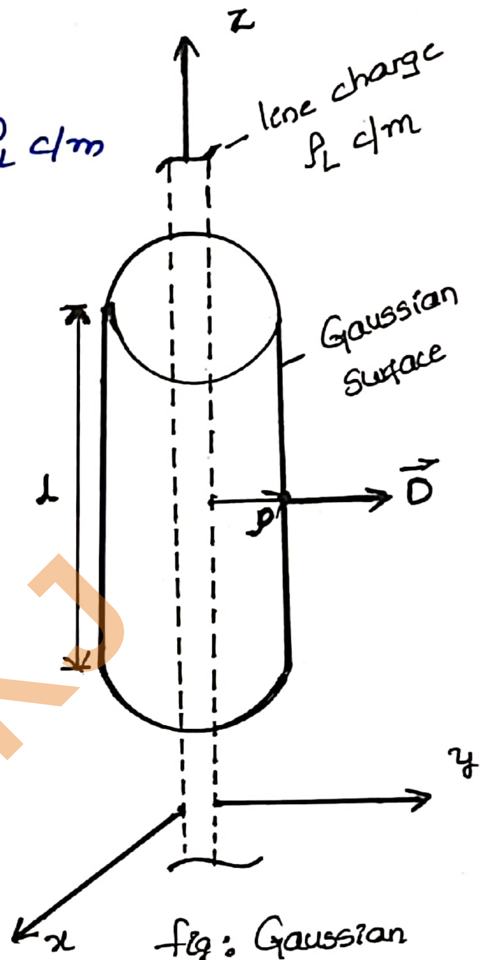


fig: Gaussian surface about an infinite line charge.

Applying Gauss Law

$$Q = \int \rho_L dL = \iint \vec{D}\cdot\vec{ds}$$

$$\iint \vec{D}\cdot\vec{ds} = \iint D_\rho \vec{a_\rho}\, \rho\, d\phi\, dz\, \vec{a_\rho}$$

$$= \int_0^{2\pi} d\phi \int_0^{l} dz\ D_\rho\cdot\rho$$

$$= D_\rho \cdot 2\pi\rho L \quad \text{— (1)}$$

and also $Q = \int \rho_L dL$ (on $dQ = \rho_L dL$)

$$Q = \rho_L l \quad (\rho_L \text{ is constant}) \quad \text{— (2)}$$

Compare the eqn's (1) & (2)

$$\rho_L \cancel{l} = D_\rho 2\pi\rho \cancel{L} \Rightarrow D_\rho = \frac{\rho_L}{2\pi\rho}$$

$$\boxed{\vec{D} = \frac{\rho_L}{2\pi\rho}\vec{a_\rho}}$$

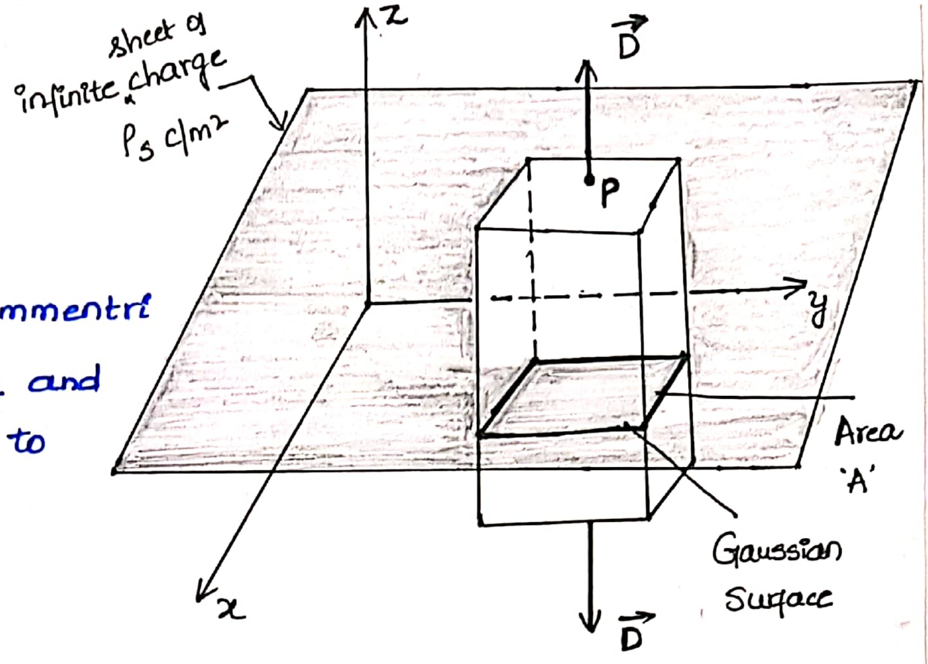
III. Infinite sheet of charge :-

consider the infinite sheet of uniform charge ρ_s c/m² lies on $\underline{z=0}$ plane (lies on x & y axis). To find $\vec{D}$ at the point 'p', the Gaussian surface is selected to be a rectangular box.

The vector $\vec{D}$ is in the positive & negetive direction

$$\boxed{\vec{D} = D_z \vec{a_z}}$$

The rectangular box is cut symmentrically by the sheet of charge and has two of its faces parallel to the sheet as shown in fig:

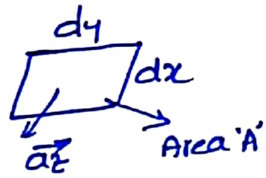


$$\int \rho_s\, ds = Q_{enc} = \iint \vec{D}\cdot\vec{ds}$$

$$= \iint_{top} D_z \vec{a_z}\cdot dxdy\, \vec{a_z} + \iint_{bottom} D_z(\vec{a_z})\, dxdy\, (-\vec{a_z})$$

$(-\vec{a_z}\cdot -\vec{a_z} = 1)$

$$= D_z \iint_{top} dxdy + D_z \iint_{bottom} dxdy$$



$$\int \rho_s\, ds = D_z\,(2A)$$

$$\rho_s(A) \Rightarrow D_z = \frac{\rho_s}{2}$$

$dxdy = A$.

$$\boxed{\vec{D} = D_z \vec{a_z} = \frac{\rho_s}{2}\vec{a_z}}$$

$$\boxed{\vec{E} = \frac{\vec{D}}{\varepsilon_0} = \frac{\rho_s}{2\varepsilon_0}\vec{a_z}}$$

$dQ = \rho_s ds$

$Q = \rho_s S$

$= \rho_s(A)$

iv: Uniformely charged sphere :-

Consider a sphere of radius 'a' with a uniform charge ρ_v c/m³.

→ To determine $\vec{D}$ everywhere we construct Gaussian surfaces for cases $r \le a$ and $r \ge a$ seperately.

→ Since the charge has spherical symmetry it is obvious that a spherical surface is an appropriate Gaussian surface.

Case 1: $r \le a$, the total charge enclosed by the spherical surface of radius 'r' as shown in fig (a)

$$Q_{enc} = \iiint \rho_v\, \vec{dv} = \rho_v \int_0^{2\pi}\int_0^{\pi}\int_0^{r} r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$\boxed{Q_{enc} = \frac{4}{3}\pi r^3 \rho_v} \quad \text{—— (1)}$$

$$\psi = \iint \vec{D}\cdot\vec{ds} = D_r \iint \vec{ds}\cdot\vec{a_r} = r^2 D_r \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta$$

$$= D_r\, 4\pi r^2 = Q_{enc} \quad \text{--- (2)}$$

compare the two eqn (1) & (2)

$$D_r\, 4\pi r^2 = 4/3\, \pi r^3 \rho_v$$

$$D_r = r/3\, \rho_v \Rightarrow \boxed{\vec{D} = \frac{r}{3}\rho_v \vec{a_r}, \quad 0 \le r \le a}$$

case 2:-

For $r \geq a$, the Gaussian surface is shown in fig (b). The charge enclosed by the surface is the entire charge in this case, i.e

$$Q_{enc} = \int \rho_v \vec{dv} = \int_0^a \int_0^{2\pi} \int_0^{\pi} \rho_v\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \rho_v \frac{4}{3}\pi a^3 \quad \text{--- (1)}$$

$$\psi = D_r\, 4\pi r^2 \quad \text{--- (2)}$$

from (1) & (2) $\quad D_r\, 4\pi r^2 = \rho_v\, 4/3\, \pi a^3$

$$D_r = \rho_v \frac{a^3}{3r^2}$$

$$\vec{D} = \rho_v \frac{a^3}{3r^2} \vec{a_r}, \quad r \geq a$$

$\vec{D}$ everywhere is given by

**

$$\boxed{\begin{aligned} \vec{D} &= r/3\, \rho_v \vec{a_r} && 0 \le r \le a \\ &= \frac{a^3}{3r^2}\rho_v \vec{a_r} && r \geq a \end{aligned}}$$

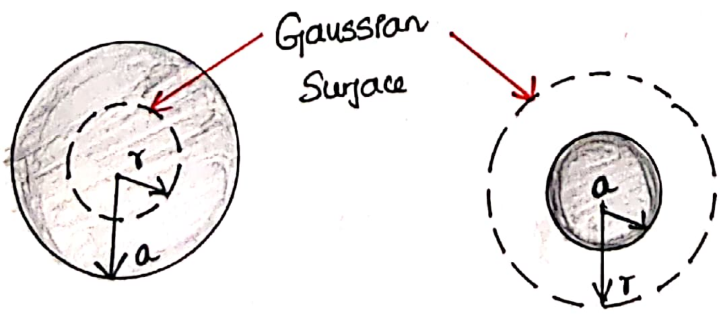
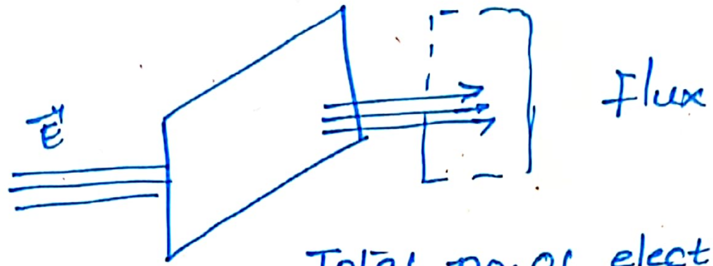


fig (a) $r \le a$ $\qquad$ fig (b) $r \geq a$

Electric flux (or) Electric displacement :-

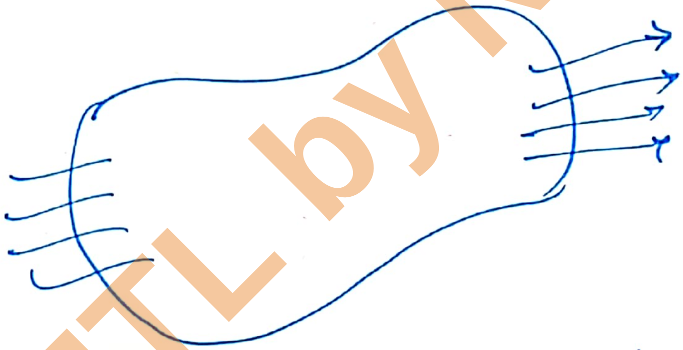
Total no. of electric Lines of force crossing the surface in a direction normal to the surface.



Total no. of electric Lines = flux

→ It is a scalar quantity and is denoted by ψ

→ is the no. of electric field Lines that pass through a surface.



According to the definition we can not calculate the flux value, because field Lines are imaginary field Lines

It is defined as $\psi = \vec{E} \cdot \vec{ds}$ → area vector

$\vec{E}$ ↓ field vector

$|\vec{ds}|$ → area of the surface

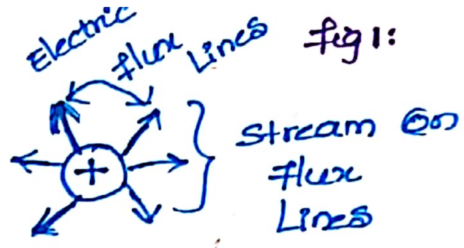
direction is very important always ⊥ar to the surface & away from it.

Electric flux :- (Michel faraday)

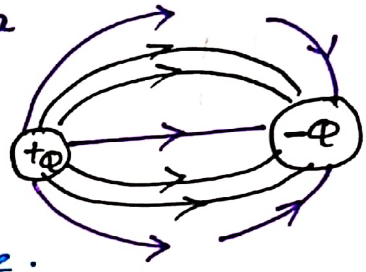
(Or)

Electric displacement :- Total no. of lines of force is called as electric flux (ψ)

→ which is scalar.
It is a scalar quantity



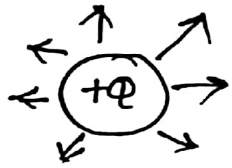
The flux lines between a +ve and -ve point charges are shown in fig: 2



Properties :-

→ * Electric flux lines always originates from +ve charge and terminate at -ve charge.

→ * If -ve charge is absent then the flux lines terminates at infinity



→ * Flux lines originates radially outwards from charged body and never cross each other.

→ * Flux lines are independent of medium

flux ψ = total Q coloumbs

→ * If the charge of the body is $\pm Q$. then

Flux line is only an imaginary line.

→ * The flux Lines from a point charge are shown in fig 1:

Electric flux ψ interms of 'D' is defined as

$$\psi = \oint_S D \cdot ds$$

Electric flux density.

Potential :-

→ A charge at rest produces potential at a specified point. It is a scalar quantity.

* Potential at a point due to a fixed charge is defined as the work done in bringing one coulomb of charge from infinity to the point against the force created by the fixed charge, i.e., the potential is the workdone per unit charge.

The potential, V at a point due to a fixed charge, 'Q_f' is given by

$$V = \frac{\text{Workdone to bring a charge 'Q' from } \infty \text{ to the point towards } Q_f}{Q}$$

Simply $V = \dfrac{\text{Workdone}}{Q}$ J/C (or) Volt

Potential difference :-

The potential difference between two points "A" and "B" is defined as the workdone by an applied force in moving a unit positive charge from "A" to "B" in electric field.

Potential difference between "A" and "B" is also defined as the difference between the potentials at "A and "B".

Equipotential Surface :

→ is one on which the potential is the same on the entire surface.

Electric potential :-

→ Suppose we wish to move a point charge Q from point A to point B in an electric field as shown in fig:

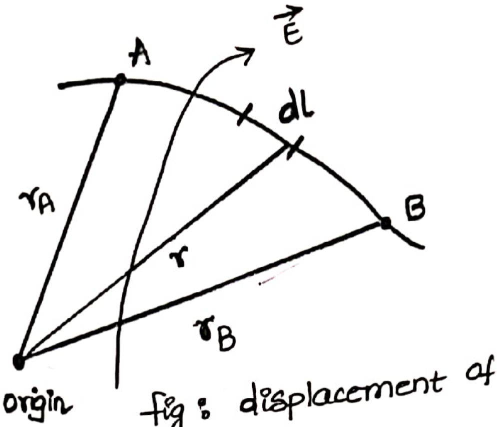


fig: displacement of point charge Q in an electrostatic field $\vec{E}$.

From Coulomb's law

$$\vec{F} = Q\vec{E}$$

So that the workdone in displacing the charge by $\vec{dl}$ is

Workdone = force × distance

$$dW = -\vec{F}\cdot\vec{dl} = -Q\vec{E}\cdot\vec{dl} \quad — ①$$

The negative sign indicates that the work is being done by an external agent on external applied force.

Thus the total workdone (on the potential energy required in moving 'Q' from A to B is

$$W = -Q\int_A^B \vec{E}\cdot\vec{dl} \quad — ②$$

divide W by Q in eqn ② gives the potential energy per unit charge. This quantity denoted by V_{AB} is known as the potential difference between points A & B. Thus

$$V_{AB} = \frac{W}{Q} = -\int_A^B \vec{E}\cdot\vec{dl} \quad \text{(Joules / coulomb)} \quad — ③$$

We are moving a point charge from A to B how much energy we spent on the system in moving a point charge from A to B (or How much workdone on a system in moving a ↓ .. . . is Called as potential.

$\vec{E}$ due to a point charge 'Q' Located at the origin, then

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\vec{a_r} \qquad V = -\int \vec{E}\cdot\vec{dl}$$

is defined as workdone by an external agent in moving a unit +ve charge from one position to another position

$$V_{AB} = -\int_{r_A}^{r_B} \frac{Q}{4\pi\varepsilon_0 r^2}\vec{a_r}\cdot dr\,\vec{a_r} = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_B} - \frac{1}{r_A}\right]$$

$$\boxed{V_{AB} = V_B - V_A} \quad — ④$$

***Potential is a scalar quantity. There is no direction.

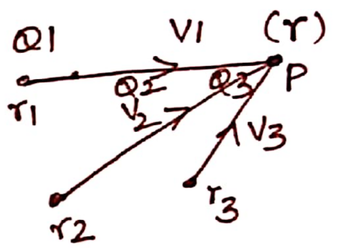
Where V_B and V_A are the potentials at B & A respectively. Thus the Potential difference V_{AB} may be regarded as the potential at 'B' with reference to 'A'.

* Work done by an external agent to move unit +ve Q from infinite to the required point.

→ Assume the potential at infinity is zero. Thus if $V_A = 0$ as $r_A \to \infty$ in eqn ④, the potential at any point ($r_B \to r$) due to a point charge Q located at the origin is

135P

$$V = V_{AB} = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r} - \frac{1}{\infty}\right] = \frac{Q}{4\pi\varepsilon_0 r} \quad — ⑤$$



Potential 'V' due to several point charges $V(r)$:-

$V = V_1 + V_2 + V_3 + \cdots + V_n$

$$V(r) = \frac{1}{4\pi\varepsilon_0}\sum_{K=1}^{n}\frac{Q_K}{|r - r_K|}$$

$$V = \frac{1}{4\pi\varepsilon_0}\frac{Q_1}{|r - r_1|} + \frac{Q_2}{4\pi\varepsilon_0|r - r_2|} + \cdots + \frac{Q_n}{4\pi\varepsilon_0|r - r_n|}$$

*** Note : Potential difference V_{AB} is independent of the path taken

Relation between $\vec{E}$ and V — Maxwell's eqn :

from fig :

$$V = V_{AB} + V_{BA} = \oint \vec{E}\cdot\vec{dl} = 0$$

$$(\because V_{AB} = -V_{BA})$$

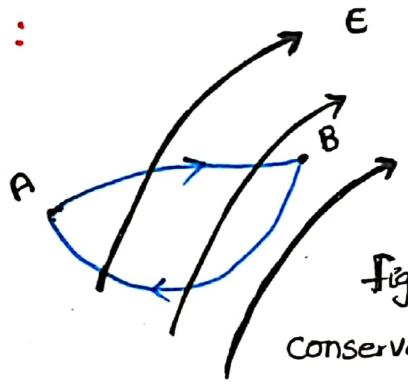


fig : Conservative nature of an electrostatic field.

$$\oint \vec{E}\cdot\vec{dl} = 0 \quad — ①$$

(This shows that the line integral of $\vec{E}$ around a closed path must be zero)

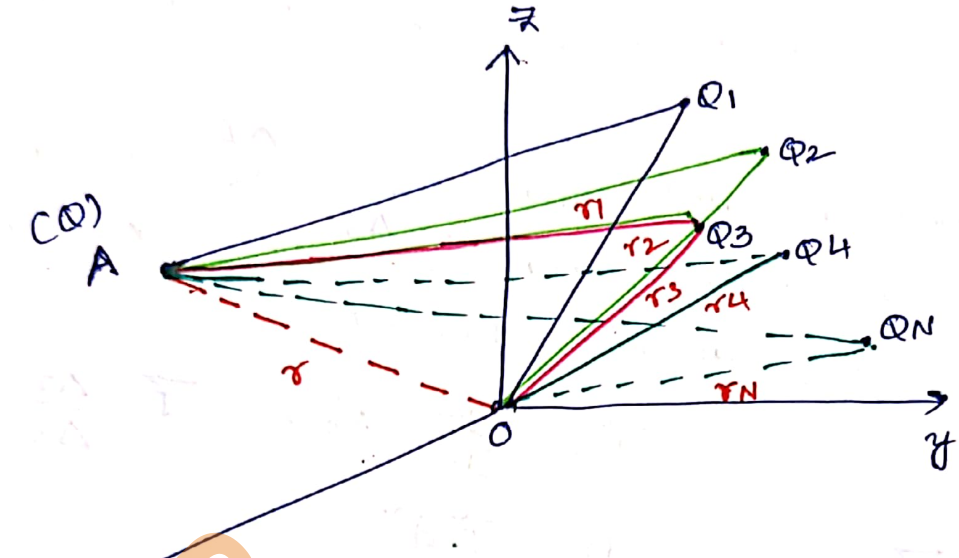
Apply Stroke's theorem

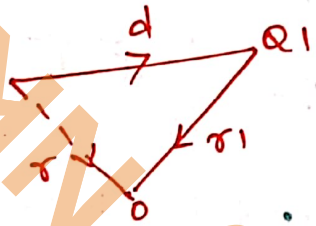
$$\oint \vec{E}\cdot\vec{dl} = \iint (\nabla\times\vec{E})\,\vec{ds}$$

from eqn ① $\iint (\nabla\times\vec{E})\,\vec{ds} = 0 \implies \boxed{\nabla\times\vec{E} = 0} \quad — ②$

Eqn ② is reffered to as Maxwell's second eqn for electrostatic fields.

Potential 'V' due to several point charges :-





$$d + r_1 - r = 0$$

$$d = r - r_1$$

$$V = V_1 + V_2 + V_3 + \cdots + V_N$$

$V_1 \rightarrow$ potential due to Q_1 at point 'A' is

$$V_1 = \frac{Q_1}{4\pi\varepsilon_0 |r - r_1|} \text{ Volts}$$

$$V = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{Q_i}{|r - r_i|} \text{ Volts.}$$

Note : Potential difference V_{AB} is independent of the path taken.

∴ the force on an electron

and also we define the potential in another way

$$V = -\int \vec{E} \cdot \vec{dl}$$

$$dv = -\vec{E} \cdot \vec{dl} = -[E_x \vec{a_x} + E_y \vec{a_y} + E_z \vec{a_z}][dx \vec{a_x} + dy \vec{a_y} + dz \vec{a_z}]$$

$$= -[E_x dx + E_y dy + E_z dz] \quad — ①$$

But from Vector caluculus

$$dv = \frac{dv}{dx} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz \quad — ②$$

Comparing the two expressions for dv

$$E_x = -\frac{\partial v}{\partial x} \; ; \; E_y = -\frac{\partial v}{\partial y} \; ; \; E_z = \frac{\partial v}{\partial z}$$

$$\vec{E} = E_x \vec{a_x} + E_y \vec{a_y} + E_z \vec{a_z} = -\left[\frac{\partial v}{\partial x} \vec{a_x} + \frac{\partial v}{\partial y} \vec{a_y} + \frac{\partial v}{\partial z} \vec{a_z} \right]$$

$$\boxed{\vec{E} = -\nabla V}$$

Convection & conduction currents :-

→ Electric voltage & current are two fundamental quantities in Electric engineering.

→ Electric current is generally caused by the motion of electric charges.

→ The current (in ampers) through a given area is the electric charge passing through the area per unit time i.e

$$\boxed{I = \frac{dQ}{dt}} \quad \text{coulombs / sec}$$

(I → amp)

Thus in a current of one amp, charge is being transfered at a rate of one Coulomb per sec.

Current density ($\vec{J}$):-

If the current ΔI flows through a surface Δs, the current density is

$$\boxed{J_n = \frac{\Delta I}{\Delta s}} \quad (\text{or}) \quad \boxed{\Delta I = J_n \Delta s}$$

assuming that the current density is ⊥ar to the surface so representing 'n' as suffix of 'J'.

→ If the current density is not normal to the surface

$$\Delta I = \vec{J}\cdot(\vec{ds})\ \Delta S$$

Thus the total current flowing through a surface 'S' is

$$\boxed{I = \int_S \vec{J}\cdot\vec{dS}}$$

Convection current :- It does not involve conductors and so it does not satisfy ohm's law. It occurs when current flows through an insulating medium such as liquid, rarefied gas (or) a vaccum.

→ A beam of electrons in a vaccum tube, for example is a convection current.

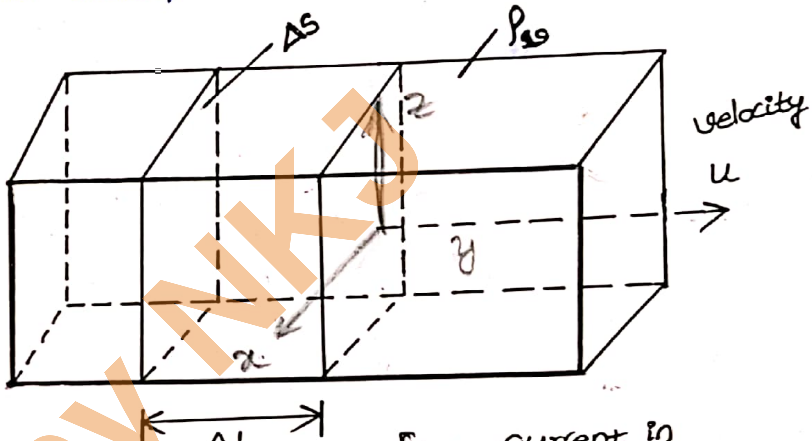


fig : Current in a filament.

consider a filament of fig :

If there is a flow of charge of density ρ_v, at velocity 'u', the current through the filament is

$$I = \frac{Q}{t} = \frac{dQ}{dt} \Rightarrow \frac{\Delta Q}{\Delta t} = \Delta I$$

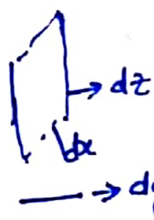
W.K.T $\quad dQ = \rho_v\, dv$

$\Delta Q = \rho_v\, \Delta V$

$$\Delta I = \rho_v \frac{\Delta V}{\Delta t}$$

$$= \rho_v \frac{\Delta S \times \Delta L}{\Delta t}$$

$$= \rho_v\, \Delta S \times u_y$$



$u = \frac{\text{distance}}{\text{time}}$

J_y (Current density along y-direction) is

$$\vec{J_y} = \frac{\Delta I}{\Delta S} = \rho_v u_y$$

The current I is the convection current and

In general $\quad \boxed{\vec{J} = \rho_v \vec{u}}$

$\vec{J}$ = convection current density in A/m^2

Conduction current :- requires a conductor. A conductor is charecterized by a large amount of free electrons that provide conduction current due to an impressed (applied) electric field.

Consider free electrons moving in a conductor under an electricfield $\vec{E}$.

When an electric field $\vec{E}$ is applied, the force on an electron with charge '-e' is $\vec{F} = Q\vec{E} = -e\vec{E}$ ($\because Q = -e$)

$$\vec{F} = -e\vec{E} = ma = \frac{m\vec{v}}{t}$$

$$\frac{m\vec{v}}{t} = -e\vec{E} \qquad \vec{v} = \frac{-e\vec{E}t}{m} \quad — ①$$

The above eqn indicates that the velocity of electron is directly proportional to the applied electric field.

$Q = -e$

$\rho_v v = -e$

$\rho_v = \boxed{-e/v}$ → one electron/unit volume

If there are 'n' electrons per unit volume, the electronic charge density is given by $\boxed{\rho_v = -ne}$

The conduction current density is

$$\vec{J} = \rho_v \vec{v} = -ne \times \frac{-et}{m}\vec{E} = \frac{ne^2t}{m}\vec{E}$$

$$* \quad \boxed{\vec{J} = \sigma\vec{E}}$$

where $\sigma = \frac{ne^2t}{m}$ is the conductivity of the conductor

Linear, Isotropic and homogeneous dielectrics :-

Linear :- A medium in which $\vec{D}$ varies linearly with $\vec{E}$ is called Linear medium $\vec{D} = \varepsilon\vec{E}$

Homogeneous :→ A medium in which ε (or) σ remains constant throughout (or) at all points is called Homogeneous medium

Isotropic :- A medium in which $\vec{D}$ & $\vec{E}$ are in the same direction. Isotropic dielectrics are those which have the same properties in all directions.

A dielectric material (in which $\vec{D} = \varepsilon\vec{E}$ applies) is linear

:→ If 'ε' does not change with the applied electric field

Homogeneous :→ If 'ε' does not change from point to point

Isotropic :→ If 'ε' does not change with direction

Continuity of current (or) Continuity Equation :

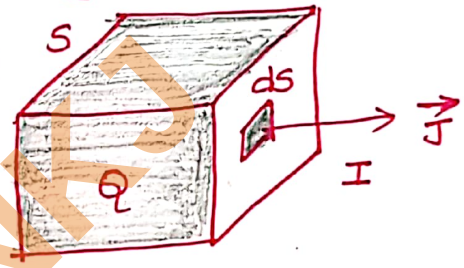
States that the current density diverging from a small volume is equal to the time rate of decrease of charge per unit volume at any given point.

$$\nabla \cdot \vec{J} = -\frac{d\rho_v}{dt} \quad \text{—— (1)}$$

Proof :-

Consider an arbitrary volume 'v' bounded by a closed surface 's' and current density $\vec{J}$ is flowing through surface area 'S'.

Let a differential surface ds be normal to the direction of current density $\vec{J}$ as shown in fig:



The total current crossing the surface is

$$I = \oint_S \vec{J} \cdot \vec{ds} \quad \text{—— (2)}$$

Let 'Q' be the charge within the closed surface. then

$$I = -\frac{dQ}{dt} \quad \text{—— (3)}$$

–ve sign indicates that the current flowing outwards.

Equating (2) & (3)

$$I = \oint_S \vec{J} \cdot \vec{ds} = -\frac{dQ}{dt}$$

From the divergence theorem

$$\oint_S \vec{J} \cdot \vec{ds} = \int_v (\nabla \cdot \vec{J}) dv \quad \text{—— (4)}$$

$$\frac{dQ}{dv} = \rho_v$$

$$dQ = \rho_v dv$$

$$Q = \int_v \rho_v dv \quad \text{—— (5)}$$

$$\int_v (\nabla \cdot \vec{J}) dv = -\frac{d}{dt} \int_v \rho_v dv$$

$$\int_v (\nabla \cdot \vec{J}) dv = \int_v -\frac{d\rho_v}{dt} dv$$

Eliminating the volume integral both sides

$$\boxed{\nabla \cdot \vec{J} = -\frac{d\rho_v}{dt}} \quad \text{—— (6)}$$

Electric current :-

* The current through a given medium is defined as charge passing through the medium per unit time. It is a scalar, i.e

$$I = \frac{dQ}{dt} \text{ , Ampere}$$

(on rate of flow of charge through a medium)

$$I = -\frac{dQ}{dt}$$

-ve sign indicates that the flow of current is opposite to the flow of $\bar{e}$'s

Current is of three types.

* Convection current
* Conduction current
* Displacement current

1. **Convection current :-** It is defined as the current produced by a beam of electrons flowing through an insulating medium. This does not obey ohm's law. For ex: current through a vaccum, liquid.

2. **Conduction current :-** It is defined as the current produced due to flow of electrons in a conductor. This obey's ohm's Law. For ex: current in a conductor Like copper.

3. **Displacement current :-** It is defined as the current which flows as a result of time-varying electric field in a dielectric material. For ex: current through a capacitor when a time-varying voltage is applied is displacement current.

a) **Current density :-** defined as the current at a given point through a unit normal area at that point. It is a vector (A/m²) denoted by "$\vec{J}$"

b) **Convection current density:** defined as the convection current at a given point through a unit normal area at that point. J_c (A/m²)

c) **Conduction current density** defined as the conduction current at a given point through a unit normal area at that point. conduction current density exists in case of conductors when an electric field is applied

d) **Displacement current density :** as the rate of displacement electric flux density with time. $J_d = dD/dt$

EMTL by NKJ

Problems 1:

1) find $\nabla\cdot\vec{A}$ at $(2,2,0)$ if $\vec{A} = \dfrac{\vec{a_x}}{\sqrt{x^2+y^2}}$

Sol :

$$\nabla\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$\vec{A} = A_x\vec{a_x} + A_y\vec{a_y} + A_z\vec{a_z}$

$\vec{A} = \dfrac{\vec{a_x}}{\sqrt{x^2+y^2}}$ $\quad \left.\begin{matrix} A_y \\ A_z \end{matrix}\right\} = 0$

$A_x = \dfrac{1}{\sqrt{x^2+y^2}}$

$$\nabla\cdot\vec{A} = \frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x^2+y^2}}\right)$$

$$= -\frac{1}{\cancel{2}}(x^2+y^2)^{-3/2}\cdot\cancel{2}x$$

$$= -(4+4)^{-3/2}\cdot 2 = \frac{-1}{8\sqrt{2}} = -0.0884$$

2) find $\nabla\cdot\vec{A}$ if $\vec{A} = \rho\sin\phi\,\vec{a_\rho} + 2\rho\cos\phi\,\vec{a_\phi} + 2z^2\vec{a_z}$

Sol : $a_\phi = a_y\cos\phi - a_x\sin\phi$; $a_\rho = \cos\phi\, a_x + \sin\phi\, a_y$, $a_z = a_z$.

③ Given that $\vec{D} = z\rho\cos^2\phi\,\vec{a_z}$ c/m^2 calculate the charge density at $(1, \pi/4, 3)$ and the total charge enclosed by the cylinder of radium 1m with $-2 \le z \le 2$ m

Sol : $\nabla\cdot\vec{D} = \rho_v$

$\vec{D} = D_\rho\vec{a_\rho} + D_\phi\vec{a_\phi} + D_z\vec{a_z}$

$a_\rho = a_\phi = 0$

$$\frac{\partial}{\partial x}D_x + \frac{\partial}{\partial y}D_y + \frac{\partial}{\partial z}D_z$$

$$\nabla\cdot\vec{D} = \frac{\partial}{\partial z}z\rho\cos^2\phi = \rho\cos^2\phi = \rho_v$$

(i) charge density ρ_v at $(1, \pi/4, 3)$ $\Rightarrow$ $\rho\cos^2\phi = 1\times\cos^2\pi/4 = 0.5$ c/m^3

(ii) charge enclosed by the cylinder $Q_{enc} = \iiint \rho_v\, d\vec{v}$

$$= \int_0^1 \rho^2 d\rho \int_0^{2\pi}\cos^2\phi\, d\phi \int_{-2}^{2} dz = \frac{\rho^3}{3}\times 2\pi\times 2 = \frac{4\pi}{3}$$

④ If $\vec{D} = (2y^2+z)\vec{a_x} + 4xy\,\vec{a_y} + x\,\vec{a_z}$ c/m^2 find (i) ρ_v (ii) ψ (iii) Q

(i) at $(-1,0,3)$; (iii) of the cube $0\le x\le 1$, $0\le y\le 1$, $0\le z\le 1$

Sol : $\rho_v = \nabla\cdot\vec{D} = 4x = 4(-1) = -4$ c/m^3

$Q = \iiint \rho_v\, d\vec{v} = 2c$

$\psi = Q = 2c$ (or) In another way $\psi = \iint \vec{D}\cdot d\vec{s}$

$$\iint(2y^2+z)\,dy\,dz + \iint 4x\,dx\,dz + \iint x\,dy\,dx = \frac{2y^3}{3}\times z + \frac{z^2}{2}\times y + \frac{4x^2}{2}z + \frac{x^2}{2}y$$

$= 13/6 = ② c$

For steady state currents $I \alpha Q \alpha \rho_v \rightarrow$ constant

$$\nabla \cdot J = -\frac{\partial}{\partial t}(\text{constant}) = 0 \Rightarrow \boxed{\nabla \cdot \vec{J} = 0} \quad (7)$$

Eq (7) shows that the total charge leaving a volume is the same as (equal to) the total charge entering it. (Kirchoff's current law follows from this)

Poisson's & Laplace equation :-

from Maxwell's 1st eqn $\nabla \cdot D = \rho_v$

$\nabla \cdot \varepsilon \vec{E} = \rho_v \Rightarrow \nabla \cdot \vec{E} = \rho_v / \varepsilon$

(ε is constant in a homogeneous medium)

$\nabla \cdot \vec{E} = \rho_v / \varepsilon$

$\nabla \cdot (-\nabla V) = \rho_v / \varepsilon \Rightarrow \boxed{\nabla^2 V = -\rho_v / \varepsilon}$

$\hookrightarrow$ Poisson's eqn in homogeneous medium

$\nabla \cdot \varepsilon \vec{E} = \rho_v \rightarrow$ Poisson's eqn in Non-homogeneous medium. (ε is not constant)

In free space there are no charges

$\rho_v = 0 \Rightarrow \boxed{\nabla^2 V = 0} \rightarrow$ which is known as Laplace equation.

Relaxation time :

The relaxation time T_r is defined as the time taken by the charge density to decay to 36.8% of its initial value at a given point. It is expressed as $\boxed{T_r = \frac{\varepsilon}{\sigma} \text{ sec}}$

where ε = Permitivity of the medium &

σ = conductivity of the "

T_r depends only on medium properties

Proof : consider a linear and homogeneous medium with constants σ and ε through which the current is flowing with current density $\vec{J}$.

W.K.T $\vec{J} = \sigma \vec{E} = \frac{\sigma \vec{D}}{\varepsilon}$ $(\vec{D} = \varepsilon \vec{E})$

X → not necessary

from Gauss Law $\nabla\cdot\vec{D} = \rho_v$

$$\nabla\cdot\varepsilon\vec{E} = \rho_v \rightarrow \nabla\cdot\vec{E} = \rho_v/\varepsilon$$

multiply bothsides $(\sigma) \rightarrow$ constant

$$\nabla\cdot\sigma\vec{E} = \sigma\rho_v/\varepsilon$$

$$\nabla\cdot\vec{J} = \sigma\rho_v/\varepsilon \quad \text{— ①}$$

from Continuity equation

$$\nabla\cdot\vec{J} = -\frac{d}{dt}\rho_v \quad \text{— ②}$$

equating ① & ②

$$\sigma\rho_v/\varepsilon = -\frac{d\rho_v}{dt}$$

(or) $$\frac{d\rho_v}{dt} + \sigma\rho_v/\varepsilon = 0 \rightarrow$$

1st order differential eqn & its solution is $\rho_v = A\,e^{\frac{-\sigma t}{\varepsilon}}$ (A → constant)

Let the initial value at $t=0$ be

By seperating Variables $\frac{d\rho_v}{\rho_v} = -\frac{\sigma}{\varepsilon}dt$

Integrating bothsides we get

$$\ln\rho_v = -\frac{\sigma}{\varepsilon}t + \ln\rho_{v0}$$

where $\ln\rho_{v0}$ is a constant of integration. Thus the above eqn is equal to

$$\rho_v = \rho_{v0}\,e^{-\sigma/\varepsilon\, t}$$

$$*** \quad \rho_v = \rho_{v0}\,e^{-t/T_r} \quad \text{where } \boxed{T_r = \frac{\varepsilon}{\sigma}}$$

At $t=0$ $\rho_v = \rho_{v0}$ is the initial charge density at $t=0$

$\rho_v = \rho_{v0}$

$\rho_v = \rho_{v0}e^{-t/T_r}$

At $t = T_r \Rightarrow \rho_v = \rho_{v0}\cdot e^{-1} = \frac{1}{e}\rho_{v0}$

$$= 0.368\,\rho_{v0}$$

∴ The relaxation time is the time taken for the (or) 1/e times charge density to reach (decay) to 36.8% ~~of the~~ its initial value.

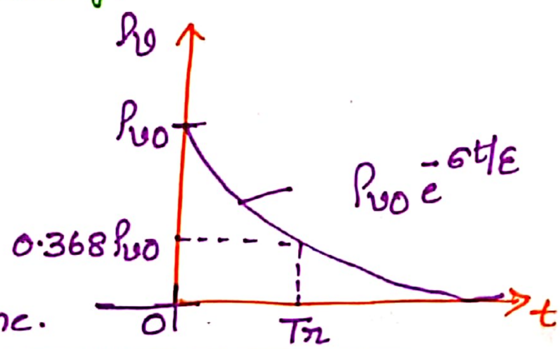


fig: Charge density Vs & relaxation time.

Capacitor :-

→ also called as condenser is an electric device having two conductors seperated by an insulator (or) dielectric medium.

→ The capacitance of a capacitor is defined as the ratio of the charge as one of its conductors to the potential difference between them. i.e

$$\boxed{C = \frac{Q}{V}} \quad \text{coulombs / volts (or) farad.}$$

Let two parallel plates in Y-Z plane be seperated by a distance 'd' meters along the x-axis. Let 'ε' be the permittivity of the dielectric medium between the plates and 'S' the surface area of the conductor

If the plates carry equal & opposite charges, an electric field develops between the plates, which can be obtained from the electric flux density as

$$\vec{D} = \rho_s \vec{a_x}$$

If $V = 1V$, $Q = 1$ coulomb then $C = 1$ Farad. Hence capacitance of a capacitor is one farad, if charge stored in one coulomb with a potential difference of one Volt.

Capacitance of Parallel plate capacitor :-

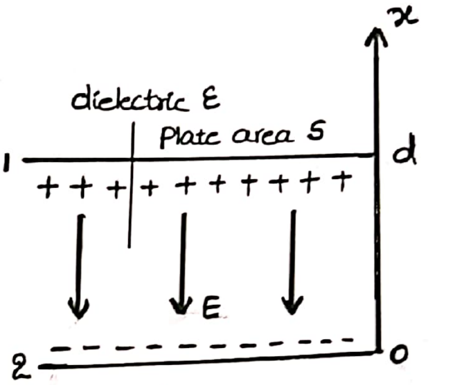


fig: Parallel plate capacitor

Consider the parallel plate capacitor of fig: Suppose that each of the plates has an area 'S' and they are seperated by a distance 'd'. Assume that plates 1 & 2 respectively carry charges $+Q$ and $-Q$ uniformly distributed on them. so that the charge density $\rho_s = Q/S$ $(\because dQ = \rho_s\, ds \rightarrow Q = \rho_s S)$

Consider 3rd Application of Gauss Law ($\vec{D}$ is along +ve z direction)

$$\vec{D} = \rho_s \vec{a_x} \rightarrow \vec{E} = \frac{\vec{D}}{\varepsilon} = \frac{\rho_s \vec{a_x}}{\varepsilon} = \frac{Q}{S\varepsilon}\vec{a_x} \quad (\rho_s = Q/S)$$

$$V = -\int \vec{E}\cdot\vec{dl} = -\int_0^d \frac{Q}{S\varepsilon}\vec{a_x}\cdot dx\,\vec{a_x} = \frac{-Qd}{S\varepsilon} = \frac{Qd}{S\varepsilon}$$

$V = \int \vec{E}\cdot\vec{dl}$

[for absolute value of 'V' ignore the negetive sign)]

$$\rightarrow C = Q/V = Q/Qd/S\varepsilon = \frac{\varepsilon S}{d} \Rightarrow \boxed{C = \frac{\varepsilon S}{d}} \text{ Farad}$$

*** The capacitance of a parallel plate capacitor $C = \frac{\varepsilon S}{d}$ Farad.

Coaxial capacitor :-

This is essentially a coaxial cable on coaxial cylindrical capacitor. Consider length l of two coaxial conductors of inner radius a and outer radius b ($b>a$) as shown in fig:

Let the space b/w the conductors be filled with a homogeneous dielectric with permittivity 'ε'. Assume that conductors ① & ② respectively carry $+Q$ & $-Q$ uniformly distributed on them.

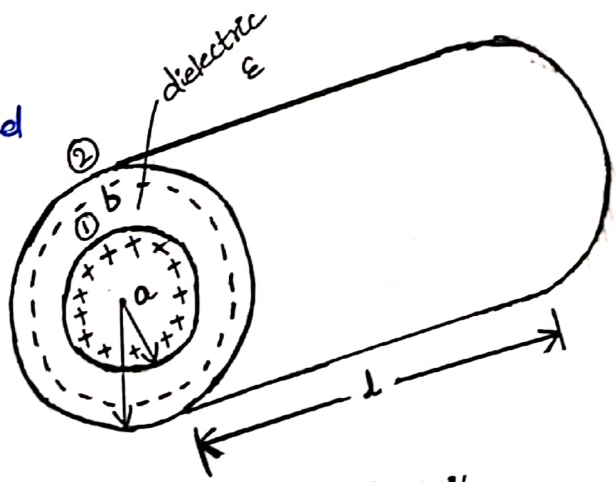


fig: Coaxial capacitor

From 2nd Application of Gauss Law

$$\vec{D} = \frac{Q}{2\pi\rho L}\vec{a_\rho} \quad ; \quad \vec{D} = \varepsilon\vec{E} \Rightarrow \vec{E} = \frac{\vec{D}}{\varepsilon} = \frac{Q}{2\pi\varepsilon\rho l}\vec{a_\rho}$$

$$V = -\int_a^b \vec{E}\cdot\vec{dl} = -\int_a^b \frac{Q}{2\pi\varepsilon\rho l}\vec{a_\rho}\cdot d\rho\,\vec{a_\rho} = -\int_a^b \frac{Q}{2\pi\varepsilon\rho l}d\rho$$

$$= \frac{-Q}{2\pi\varepsilon l}\left[\ln\rho\right]_a^b = \frac{Q}{2\pi\varepsilon l}(\ln b - \ln a)$$

neglect -ve sign for absolute value of 'V'

$$C = \frac{Q}{V} = \frac{Q}{Q/2\pi\varepsilon l\,(\ln b - \ln a)} = \frac{2\pi\varepsilon l}{\ln(b/a)}$$

Capacitance of a Coaxial capacitor $\boxed{C = \frac{2\pi\varepsilon l}{\ln(b/a)} \text{ Farads}}$

Spherical Capacitor :-

From 1st Application of Gauss Law

$$\vec{D} = \frac{Q}{4\pi r^2}\vec{a_r} \qquad \vec{D} = \varepsilon\vec{E}$$

$$\vec{E} = \frac{\vec{D}}{\varepsilon} = \frac{Q}{4\pi\varepsilon r^2}\vec{a_r}$$

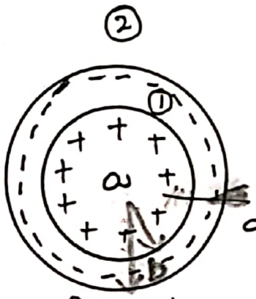


fig: Spherical capacitor

$$V = -\int_a^b \vec{E}\cdot\vec{dl} = -\int_a^b \frac{Q}{4\pi\varepsilon r^2}\vec{a_r}\cdot dr\,\vec{a_r}$$

$$= \frac{+Q}{4\pi\varepsilon}\left[\frac{1}{b} - \frac{1}{a}\right]$$

neglect -ve sign

$$= \frac{Q}{4\pi\varepsilon}\left[\frac{1}{a} - \frac{1}{b}\right]$$

$$C = \frac{Q}{V} = \frac{Q}{Q/4\pi\varepsilon\left(\frac{1}{a} - \frac{1}{b}\right)}$$

$$= \frac{4\pi\varepsilon}{\frac{1}{a} - \frac{1}{b}}$$

The capacitance of a Spherical capacitor $\boxed{C = \frac{4\pi\varepsilon}{1/a - 1/b} \text{ Farads.}}$

Consider the inner sphere of radius a and outer sphere of radius 'b' seperated by a dielectric medium with permitivity ε as shown in fig.

4) ∇×H = J ... Maxwell's 3rd eq

Electrostatic Energy density :-

Amount of energy stored in a capacitor can be found by calculating the workdone in charging a capacitor.

$$\text{Potential} = \frac{\text{Workdone}}{\text{unit charge}} \Rightarrow V = \frac{W}{Q} \Rightarrow W = VQ$$

Workdone in moving a small charge dq against a potential difference V is

$$dw = V\,dq \qquad Q = CV, \quad V = Q/C$$

$$dw = Q/C\,dq$$

Total workdone in charging a capacitor to Q coulombs is

$$w = \int_0^Q \frac{Q}{C}\,dq = \frac{Q}{C}\times\frac{Q}{2} = \frac{Q^2}{2C} = \frac{1}{2}CV^2$$

Electrostatic energy stored $\boxed{w = \frac{Q^2}{2C} = \frac{1}{2}CV^2} = \frac{1}{2}QV$ — for one point charge

Energy stored in a capacitor :-

$$w = \frac{1}{2}CV^2 \Rightarrow \Delta w = \frac{1}{2}\Delta C(\Delta V)^2 \qquad C = \frac{\varepsilon(A=S)}{d}$$

$$\Delta w = \frac{1}{2}\varepsilon\frac{\Delta S}{\cancel{\Delta d}}\times E^2\times \Delta d^{\cancel{2}} \qquad \Delta C = \frac{\varepsilon\Delta A(\Delta S)}{\Delta d}$$

$$= \frac{1}{2}\varepsilon E^2\,\Delta S\times\Delta d = \frac{1}{2}\varepsilon E^2\Delta v \qquad E = \frac{V}{d}$$

$$w = \int \frac{1}{2}\varepsilon E^2 dv = \frac{1}{2}\int \vec{D}\cdot\vec{E}\,dv \qquad \frac{\Delta V}{\Delta d} = E$$

$E \to$ Static electric field

Electrostatic Energy density w_E (in J/m^3) as

$$w_E = \frac{d}{dv}(w) = \frac{d}{dv}\int\frac{1}{2}\varepsilon E^2 dv$$

$$= \boxed{\frac{1}{2}\varepsilon \vec{E}^2 \text{ (or) } \frac{1}{2}\vec{D}\cdot\vec{E}}$$

*** Electrostatic Energy due to n point charges is

$$\boxed{W_E = \frac{1}{2}\sum_{K=1}^{n} Q_K V_K}$$

SUMMARY OF UNIT-1

① The two fundamental Laws for electrostatic fields

1> Coulomb's Law 2> Gauss Law

Coulomb's law :- $\vec{F} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 r^2} \vec{a_r}$ Newtons

Gauss Law : The net electric flux (passes) Penetrating a closed surface is equal to the total charge enclosed. $\psi = \oint \vec{D} \cdot \vec{ds} = Q_{enc} = \int \rho_v dv$

② Based on Coulomb's Law, the electric field intensity $\vec{E}$ (or electric field strength)

$$\vec{E} = \frac{\vec{F}}{Q} = \frac{Q}{4\pi\varepsilon_0 r^2} \vec{a_r} \quad (Q_1 = Q_2 = Q) \quad N/C \text{ (or } V/m)$$

③ For a continueous charge distribution, the total charge is given by

$Q = \int \rho_L \, dL$ for line charge

$Q = \int \rho_s \, ds$ for surface charge

$Q = \int \rho_v \, dv$ for volume charge.

④. Based on Faraday's experiment, electric displacement is equal to the charge enclosed. $\boxed{\psi = Q}$ coulombs

⑤ The electric flux density $\vec{D}$ is related to the electric field intensity (in freespace) as

$$\boxed{\vec{D} = \varepsilon_0 \vec{E}}$$ (or electric displacement density)

The electric flux through a surface S is → $$\boxed{\psi = \int_S \vec{D} \cdot \vec{ds}}$$ C/m²

Electrostatic Energy density :-

⑥ From Gauss Law $\boxed{\nabla \cdot \vec{D} = \rho_v}$ → Maxwell's 1st equation

⑦ Applications of Gauss Law

Gaussian surface is the

1) Point charge : $\vec{D} = \dfrac{Q}{4\pi r^2}\vec{a_r}$ → sperical surface

↓

2) Infinite line charge : $\vec{D_\rho} = \dfrac{\rho_L}{2\pi\rho}\vec{a_\rho}$ → cylindrical surface

↓

3) Infinite sheet of charge :

$\vec{D} = \rho_s/2 \, \vec{a_z}$ → Rectangular box

4) uniformly charged sphere :

$$\vec{D} = \frac{r}{3}\rho_v \vec{a_r} \qquad 0 \le r \le a$$

$$= \frac{a^3}{3r^2}\rho_v \vec{a_r} \qquad r \ge a$$

⑧ Force due to several point charges

$$\vec{F} = \frac{Q}{4\pi\varepsilon_0}\sum_{K=1}^{N}\frac{Q_K(r - r_K)}{|r - r_K|^3}$$

$$\vec{E} = \frac{\vec{F}}{Q} = \frac{1}{4\pi\varepsilon_0}\sum_{K=1}^{N}\frac{Q_K(r - r_K)}{|r - r_K|^3}$$

⑨ The total workdone, or the electric potential energy to move a point charge from point A to point B in electric field $\vec{E}$ is

$$W = -Q\int_A^B \vec{E}\cdot\vec{dl} \qquad \text{Joules}$$

⑩ The potential difference V_{AB}, the potential at B with reference to A is

$$V_{AB} = -\int_A^B \vec{E}\cdot\vec{dl} = \frac{W}{Q} = V_B - V_A \qquad \text{Jouls/coulomb (or) Volts}$$

⑪ The potential at r due to a point charge Q at r' is

↓ due to several point charges

$$V(r) = \frac{1}{4\pi\varepsilon_0}\sum_{K=1}^{n}\frac{Q_K}{|r - r_K|}$$

(12) Since ~~an~~ electric field is conservative (the net Workdone along a closed path in a static $\vec{E}$ field is zero)

i.e $\oint \vec{E} \cdot \vec{dl} = 0$

from that $\boxed{\nabla \times \vec{E} = 0}$ $\rightarrow$ Maxwell's 2nd equation

(13) Given the potential field, the corresponding electric field is found to be $\boxed{\vec{E} = -\nabla V}$

(14) The electric current density through a surface's i.e electric current is the flux of

$$\boxed{I = \int \vec{J} \cdot \vec{ds}}$$

(15) Convection current occurs when the current flows through an insulating medium such as gas, liquid or vaccum.

$$\boxed{\vec{J} = \rho_v \vec{u}}$$

$\downarrow$

Convection current density

(16) Conduction current density $\boxed{\vec{J} = \sigma \vec{E}}$

σ $\downarrow$ Conductivity of the conductor

(17) a) <u>Linear</u> :– $\vec{D}$ varies linearly with $\vec{E}$

b) <u>homogeneous</u> :- ε or σ remains constant

c) <u>Isotropic</u> :– $\vec{D}$ & $\vec{E}$ are in the same direction

(18) From continuity equation $\boxed{\nabla \cdot \vec{J} = -\frac{d\rho_v}{dt}}$

For steady state currents $\rho_v = 0$ $\Rightarrow$ $\boxed{\nabla \cdot \vec{J} = 0}$

(19) Relaxation time or Rearrangement time $\boxed{T_r = \frac{\varepsilon}{\sigma}}$

4> $\nabla \times \vec{H} = \vec{J}$

(20) In a homogeneous medium

Poisson's eqn $\boxed{\nabla^2 V = -\rho_v/\varepsilon}$

If there are no charges in free space $\boxed{\nabla^2 V = 0}$

↓ Laplace eqn

(21) In a non-homogeneous medium

$\boxed{\nabla \cdot \varepsilon \nabla V = -\rho_v}$ → Poisson's eqn

(22) capacitance of a parallel plate capacitor $\Rightarrow C = \frac{\varepsilon S}{d}$

Coaxial capacitor $\Rightarrow C = \frac{2\pi\varepsilon L}{\ln(a/b)}$ F

Spherical capacitor $\Rightarrow C = \frac{4\pi\varepsilon}{-\frac{1}{a}+\frac{1}{b}}$ F

(23) The electrostatic energy due to n point charges is

$$\boxed{W_E = \frac{1}{2}\sum_{k=1}^{n} q_k V_k}$$

For a continuous volume charge distribution

$$W = W_E = \frac{1}{2}\int_v \vec{D}\cdot\vec{E}\,dv = \frac{1}{2}\int \varepsilon_0 |E|^2 dv$$

(24) The electrostatic energy density $\boxed{w_E = \frac{1}{2}\varepsilon E^2 = \frac{1}{2}\vec{D}\cdot\vec{E}}$ (J/m^3)

(25) Boundary conditions:

dielectric - dielectric interface $\boxed{E_{1t} = E_{2t}}$

$\boxed{D_{1n} - D_{2n} = \rho_s \text{ (or) } D_{1n} = D_{2n} \text{ if } \rho_s = 0}$

For a dielectric - conductor interface

$\boxed{E_t = 0}$, $\boxed{D_n = \varepsilon E_n = \rho_s}$

because $\vec{E} = 0$ inside the conductor.

Subjective Exam - II

① define and distinguish between the terms $\perp$ar & $\parallel$el polarization for the case of reflection by a perfect conductor under oblique incidence.

② determine the resultant electric and magnetic fields of plane wave when it is incident on a perfect conductor normally.

③ Write short notes on the following

(a) Surface impedance (b) Brewster angle

(c) critical angle (d) Total internal reflection

④ Explain the significance of poynting theorem & poynting vector.

⑤ Assuming z-direction of propagation in a parallel plane W/G. determine the expressions for the transverse field components in terms of partial derivatives of E_z and H_z.

⑥ derive the Relation $\lambda = \dfrac{\lambda_c \lambda_g}{\sqrt{\lambda_g^2 + \lambda_c^2}}$

⑦ define, derive & explain the significance of the following terms as applicable to parallel plane guides

(a) Wave impedance (b) phase & group velocities

(c) cut-off-freq

⑧ Explain about attenuation in parallel plate W/G's also draw attenuation versus freq characteristics of wave guided between parallel conducting plates.

9. Explain the different types of Tx-lines. What are the limitations to use max-power that they can handle.

10. derive the characteristic impedance of a Tx-line in terms of its line constants.

11. Starting from the equivalent ckt, derive the tx-line equations for V & I in terms of the source parameters.

12. Explain about the infinite line

13. Explain the conditions which are used for minimum attenuation on distortion less transmission

14. What is Loading ? Explain the different types of Loading in Tx-lines.

15. define the reflection coefficient and derive the expression for input impedance in terms of reflection coefficient.

16. Explain the principle of impedance matching with $\lambda/4$ transformer.

17. Explain the properties & Applications of Smith chart.

18. Explain the significance & design of single stub impedance matching. discuss the factors on which stub length depends.

A long straight conductor with radius 'a' has a magnetic field strength $H = \frac{Ir}{2\pi a^2}\vec{a_\phi}$ with in the conductor $(r<a)$ and $H = \frac{I}{2\pi r}\vec{a_\phi}$ outside the conductor $(r>a)$. Find the current density "J" in both the regions $(r<a \,\&\, r>a)$

Sol: $\nabla \times H = J$

$\nabla \times H$ in cylindrical co-ordinate system

$$\nabla \times H = \frac{1}{\rho}\begin{vmatrix} \vec{a_\rho} & \vec{a_\phi} & \vec{a_z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ H_\rho & \rho H_\phi & H_z \end{vmatrix}$$

given data $H = H_\rho\vec{a_\rho} + H_\phi\vec{a_\phi} + H_z\vec{a_z} = \frac{Ir}{2\pi a^2}\vec{a_\phi}$

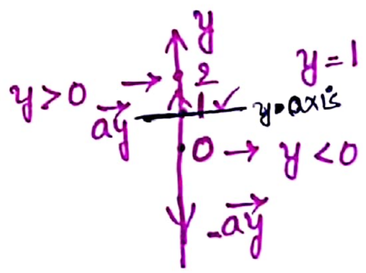
$$H_\phi = \frac{Ir}{2\pi a^2}$$

$$\nabla \times H = \frac{1}{\rho}\begin{vmatrix} \vec{a_\rho} & \vec{a_\phi} & \vec{a_z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & \frac{I\rho^2}{2\pi a^2} & 0 \end{vmatrix}$$

$$= \frac{I}{\pi a^2}\vec{a_z} \text{ A/m}^2 \text{ for } r<a$$

$$= 0 \text{ for } r>a.$$

2] A conducting plane at $y=1$ carries a surface current of $10\vec{z}$ mA/m. Find H & B at (0,0,0) & at (2,2,2).



Sol $\vec{H} = \frac{1}{2}\vec{k} \times \vec{a_n}$

$$\vec{H}_{(0,0,0)} = \frac{1}{2} \times 10\vec{z} \times (-\vec{y})$$

$$= \frac{1}{2} \times 10 \times \vec{a_x} \text{ mA/m} = 5\vec{a_x} \text{ mA/m}$$

$$\vec{H}_{(2,2,2)} = \frac{1}{2} \times 10\vec{a_z} \times (\vec{a_y}) \text{ mA/m}$$

$$= -5\vec{a_x} \text{ mA/m}$$

$\vec{B}(0,0,0) = \mu_0 \times 5\vec{a_x}$ mA/m, & $\vec{B}(2,2,2) = -\mu_0 5\vec{a_x}$ mA/m

3] X-Z plane is a boundary between two dielectrics. Region $y<0$ contains dielectric material $\epsilon_{r_1} = 2.5$ while region $y>0$ has dielectric with $\epsilon_{r_2} = 4.0$. If $\vec{E} = -30\vec{a_x} + 50\vec{a_y} + 70\vec{a_z}$ V/m. find normal & tangential components of the $\vec{E}$ field on both sides of the boundary.

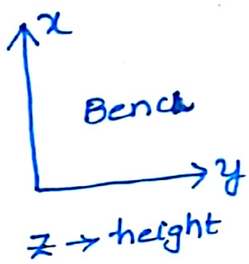
Ans Assume $\vec{E}$ belongs to medium 1.

$$\vec{E_1} = -30\vec{a_x} + 50\vec{a_y} + 70\vec{a_z} \text{ V/m} \qquad \vec{E} = \vec{E_t} + \vec{E_n}$$

$$\vec{E_1} = \vec{E_{1t}} + \vec{E_{1n}} \Rightarrow E_{1t} = -30\vec{a_x} + 70\vec{a_z} \text{ V/m}$$

$$E_{1n} = 50\vec{a_y}$$

W.K.T $\boxed{E_{1t} = E_{2t}}$ $\boxed{E_{2n} = \dfrac{\epsilon_{r1} E_{1n}}{\epsilon_{r2}}}$



X & Z are parallel &
Y is Perpendicular.

X-Z is a boundary

UNIT - II

MAGNETOSTATICS

By,

N. Krishna Jyothi
ECE - GNITS

References
1. No sadiku

UNIT – 2

UNIT-2

Magnetostatics:

→ An electrostatic field is produced by static (or) stationary charges

→ If the charges are moving with constant velocity, a static magnetic field (or) magnetostatics) is produced.

→ A magnetostatic field is produced by a constant current flow (or direct current).

→ This current flow may be due to magnetization currents as in permanent magnets (or) conduction currents as in current carrying wires.

Analogies :-

Electrostatics			Magnetostatics	
1>	$\vec{E}$	Volts/meter	$\vec{H}$	Amp/meter
2>	ψ	coulomb	ψ (or) ϕ	webers
3>	$\vec{D}$	coulomb/meter^2	$\vec{B}$	webers/m^2
4>	ε	Farad/m	μ	Henry/m
5>	V	Volts	I	amps
6>	$\vec{D} = \varepsilon\vec{E}$		$\vec{B} = \mu\vec{H}$	
7>	$V = \int \vec{E}\cdot\vec{dl}$		$I = \int \vec{H}\cdot\vec{dl}$	
from ohm's Law	$V = El = IR$		from ohm's Law	$Hl = I$

There are two major Laws in magnetostatics fields

1> Biot-Savart's Law   II > Ampere's Law

Similar to Coulomb's law and Gauss' law in Electrostatic fields.

Biot-Savart's law

→ States that the magnetic field Intensity $\vec{dH}$ Produced at a Point P, as shown in figure. by the differential current element Idl is Proportional to the Product Idl and the sine of the angle 'α' between the element and the line joining P to the element and is inversely proportional to the square of the distance 'R' between P and the element

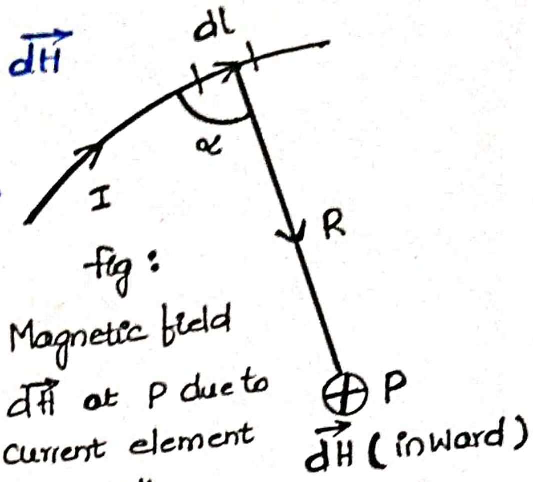


fig: Magnetic field $\vec{dH}$ at P due to current element Idl

i.e $\quad \vec{dH} \propto \dfrac{Idl \sin\alpha}{R^2}$ — ①

$\vec{dH} = \dfrac{K \times Idl \sin\alpha}{R^2}$ — ② $\quad$ Where K is the constant of proportionality $= \dfrac{1}{4\pi}$

$\vec{dH} = \dfrac{Idl \sin\alpha}{4\pi R^2} = \dfrac{I\ \vec{dl} \times \vec{a_R}}{4\pi R^2}$

$= \dfrac{I\ \vec{dl} \times \vec{R}}{4\pi R^3}$ — ③

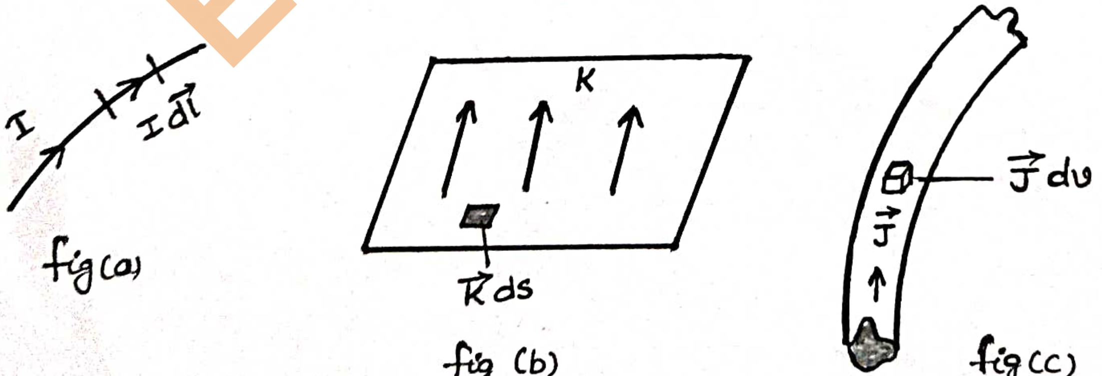
Cross product of $\vec{dl} \times \vec{a_R} = |dl||a_R| \sin\alpha$

$= dl \sin\alpha$ (magnitude of unitvector $=1$)

where $R = |R|$ and $\vec{a_R} = \vec{R}/|R|$

In electrostatic fields there are different types of charge distributions ($\rho_L \vec{dl}$, $\rho_S \vec{ds}$, $\rho_v \vec{dv}$)

III'ly magnetostatic fields have different current distributions



fig(a) $\quad$ fig (b) $\quad$ fig(c)

Figure: Current distributions

(a) Line current $\quad$ (b) Surface current $\quad$ (c) Volume current

$$I\,\vec{dl} = \vec{K}ds = \vec{J}dv$$

Where $\vec{K}$ is the surface current density (amp/meter) and $\vec{J}$ is volume current density (amp/m²)

The distributed current sources are represented using Biot-Savart's law

$$\vec{H} = \int_L \frac{I\,\vec{dl} \times \vec{a_R}}{4\pi R^2} \quad \text{(Line current)}$$

$$\vec{H} = \int_S \frac{\vec{K}\,ds \times \vec{a_R}}{4\pi R^2} \quad \text{(surface current)}$$

$$\vec{H} = \int_v \frac{\vec{J}\,dv \times \vec{a_R}}{4\pi R^2} \quad \text{(volume current)}$$

Magnetic field intensity $\vec{H}$ due to conductor of Infinite length :-

→ The conductor is along the z-axis with its upper and lower ends respectively subtending angles α_2 & α_1 at P, the point at which $\vec{H}$ is to be determined.

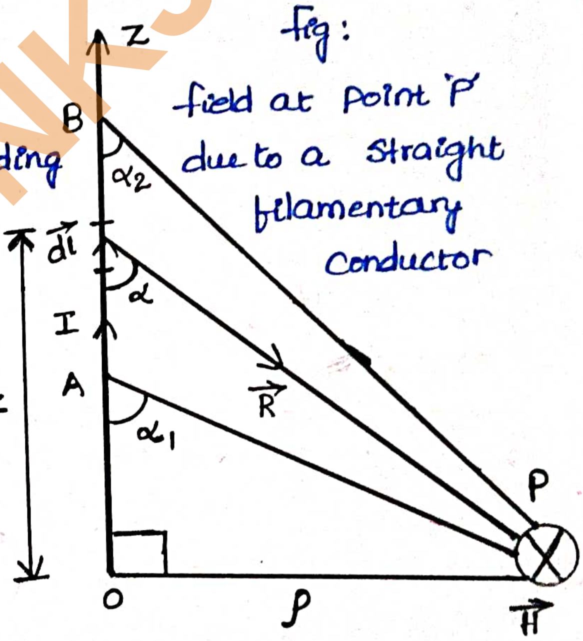


fig: field at point 'P' due to a straight filamentary conductor

→ Consider the contribution $\vec{dH}$ at 'P' due to an element $\vec{dl}$ at (0,0,z)

$$\vec{R} = (0, \rho, 0) - (0, 0, z)$$

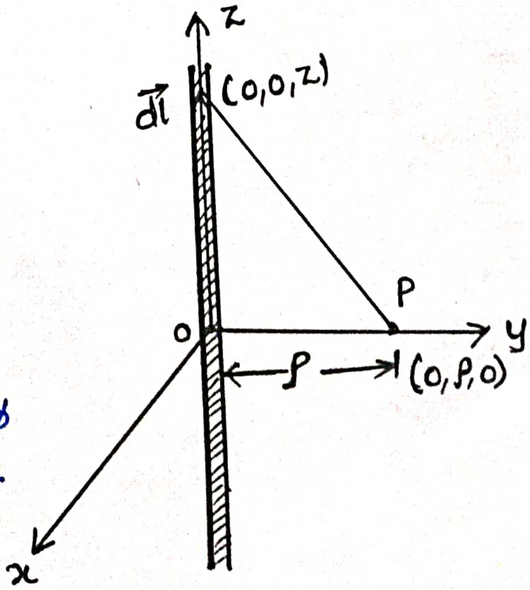
$$= \rho\vec{a_y} - z\vec{a_z}$$

In cylindrical $\vec{R} = \rho\vec{a_\rho} - z\vec{a_z}$

But $\vec{dl} = dz\,\vec{a_z}$ and $\vec{R} = \rho\vec{a_\rho} - z\vec{a_z}$

$$\vec{dl} \times \vec{R} = dz\,\vec{a_z} \times (\rho\vec{a_\rho} - z\vec{a_z})$$

$$= \rho\,dz\,\vec{a_\phi} \quad \& \quad |R| = \sqrt{\rho^2 + z^2}$$



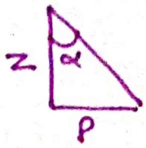
Hence $$\vec{H} = \int \frac{I\,\vec{dl} \times \vec{R}}{4\pi R^3} = \int \frac{I\rho\,dz}{4\pi(\rho^2 + z^2)^{3/2}}\,a_\phi$$

⊙ $\vec{H}$ outwards ⊗ $\vec{H}$ inwards

Ampere's law is applied to this path

$z = \rho \cot\alpha$, $\frac{dz}{d\alpha} = \rho(-\mathrm{cosec}^2\alpha)$ from fig:

$dz = -\rho\, \mathrm{cosec}^2\alpha \, d\alpha$



Substitute in the above equation we get

$$\vec{H} = \int \frac{-I \rho^2 \mathrm{cosec}^2\alpha \, d\alpha}{4\pi(\rho^2 + \rho^2\cot^2\alpha)^{3/2}} a_\phi = \frac{-I}{4\pi}\int_{\alpha_1}^{\alpha_2} \frac{\rho^2 \mathrm{cosec}^2\alpha \, d\alpha}{\rho^3 \mathrm{cosec}^3\alpha} a_\phi$$

$$\vec{H} = \frac{-I}{4\pi}\int_{\alpha_1}^{\alpha_2} \frac{1}{\rho}\sin\alpha \, d\alpha \, a_\phi = \frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1) a_\phi$$

Case 1: Substitute $\alpha_1 = 90°$; $\alpha_2 = 0°$ → When the conductor is semi finite (w.r.to P) so that A is now at O (0,0,0) while B is at (0,0,α) ;

$$\boxed{\vec{H} = \frac{I}{4\pi\rho} a_\phi}$$

$\rho \cot\alpha_1 = 0$
$\alpha_1 = \cot^{-1}(0)$
$= 90°$

case 2: $\alpha_1 = 180°$ & $\alpha_2 = 0°$ → $\boxed{\vec{H} = \frac{I}{2\pi\rho} a_\phi}$

conductor is infinite in length. For this case A is at (0,0,−α) while B is at (0,0,α)
↓ $\alpha_2 = 0°$

$\rho\cot\alpha_2 = \infty$
$\alpha_2 = \cot^{-1}(\infty)$
$= 0°$

Ampere's Circuit Law – Maxwell's eqn :

→ States that the line integral of the tangential component of $\vec{H}$ around a closed path is the same as the net current I_{enc} enclosed by the path.

In otherwords the circulation of $\vec{H}$ equals I_{enc} i.e

*** $$\boxed{\oint_L \vec{H}\cdot\vec{dl} = I_{enc}}$$

By applying Stroke's theorem

$$\oint_L \vec{H}\cdot\vec{dl} = \iint_S (\nabla\times\vec{H})\cdot\vec{ds} = I_{enc}$$

W.K.T $I = \iint_S \vec{J}\cdot\vec{ds} = \iint_S (\nabla\times\vec{H})\cdot\vec{ds} \Rightarrow \boxed{\nabla\times\vec{H} = \vec{J}}$ → This is the 3rd Maxwell eqn

Applications of Ampere's law

A > Infinite Line Current :-

Consider an infinitely long filamentary current 'I' along the z-axis as in fig. To determine $\vec{H}$ at an observation point 'P'. Amperian Path is used to determine $\vec{H}$ (similar to Gaussian surface in Electric fields)

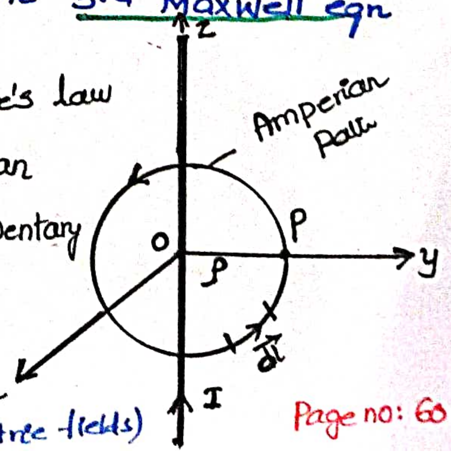


fig: Ampere's law applied to an infinite filamentary line current in the direction of a_ϕ

Ampere's law is applied to this path

$$I = \oint_L \vec{H} \cdot \vec{dl}$$

$\vec{H}$ is along a_ϕ direction is $H_\phi a_\phi$

$\vec{dl}$ along a_ϕ direction is $\rho\, d\phi\, a_\phi$

$$I = \oint_0^{2\pi} H_\phi a_\phi \, \rho\, d\phi\, a_\phi$$

$$\Rightarrow H_\phi \, 2\pi\rho = I \Rightarrow H_\phi = \frac{I}{2\pi\rho}$$

$$\vec{H} = H_\phi a_\phi = \boxed{\frac{I}{2\pi\rho} a_\phi}$$

B> Infinite Sheet of Current :-

Consider an infinite current sheet in the $z=0$ plane. If the sheet has a uniform current density $\vec{K} = k_y \vec{a_y}$ and the current is flowing in the y-direction.

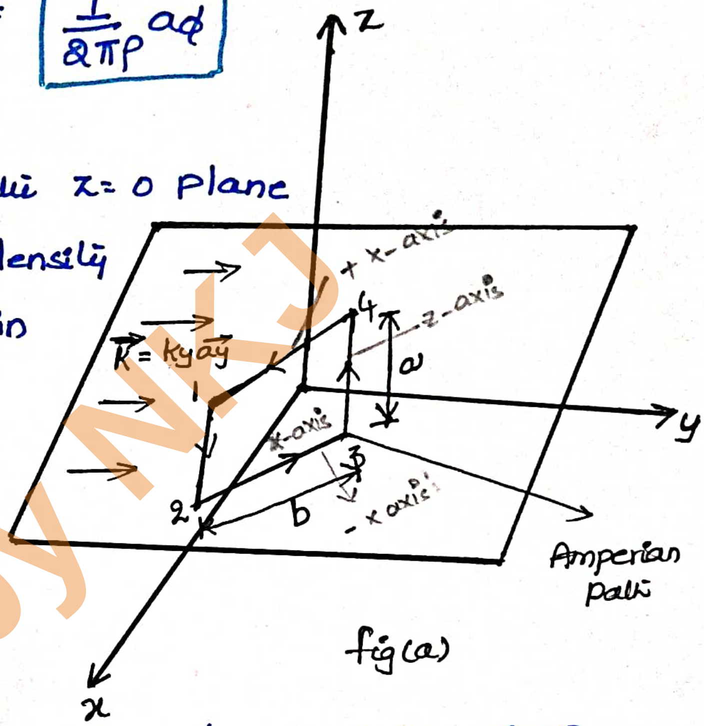


fig (a)

The surface current density $\vec{K}$ is defined as current per unit width

$$k_y = I/b \quad A/m$$

$$\Rightarrow I = k_y b$$

Applying Ampere's law to the rectangular closed path gives

$$\oint \vec{H} \cdot \vec{dl} = I_{enc} = k_y b$$

A closed amperian loop 12341 is selected of width 'b' and height 'a' such that it is ⊥ar to the flow of current (current flowing in the y-direction) (loop is along x & z-axis)

*** Assume that $\vec{H}$ is having only x-component and is defined as

tangent
Straight line

$$\vec{H} = \begin{cases} H_0 \vec{a_x} & z>0 \\ -H_0 \vec{a_x} & z<0 \end{cases}$$

and also it is tangential to the paths $2 \to 3$ and $4 \to 1$

$$\oint \vec{H} \cdot \vec{dl} = \left(\int_1^2 + \int_2^3 + \int_3^4 + \int_4^1 \right) \vec{H} \cdot \vec{dl} = 0(-a) + (-H_0)(-b) + 0(a) + H_0(b)$$

$$k_y b = 2H_0 b \Rightarrow \boxed{H_0 = \frac{1}{2} k_y}$$

$$\vec{H} = \begin{cases} \frac{1}{2} k_y \vec{a_x} & z>0 \\ -\frac{1}{2} k_y \vec{a_x} & z<0 \end{cases}$$

In general for an infinite sheet of current density K A/m, $\boxed{\vec{H} = \frac{1}{2} \vec{K} \times \vec{a_n}}$

Where $\vec{a_n}$ is a unit vector normal directed from the current sheet to the point 'P'

Differential or Point form   Integral form

Magnetic flux density - Maxwell's equation

→ The magnetic flux density $\vec{B}$ is related to the magnetic field Intensity $\vec{H}$ according to

$$\boxed{\vec{B} = \mu_0 \vec{H}}$$

Similar to $\vec{D} = \varepsilon_0 \vec{E}$

Where μ_0 is a constant known as the Permeability of freespace (H/m)

$$\mu_0 = 4\pi \times 10^{-7} \text{ H/m}$$

→ The magnetic flux through a surface 'S' is given by

$$\boxed{\psi = \int_S \vec{B} \cdot d\vec{s}}$$

$\psi = \int_S \vec{B} \cdot d\vec{s}$

→ The magnetic flux lines due to a straight long wire as shown in fig: The direction of $\vec{B}$ (magnetic flux density) is taken as that indicated as north by the needle of magnetic compass.

→ Note that each flux line is closed and has no beginning or end. For a straight current carrying conductor; it is generally true that magnetic flux lines are closed and do not cross each other regardless of current distribution.

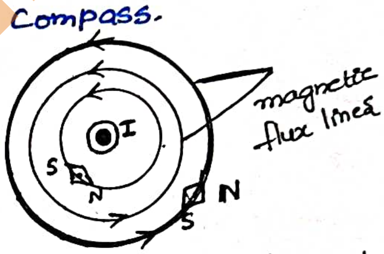


fig: Magnetic flux lines due to a straight wire with current coming out of the page.

→ The charge Q is the source of the lines of the electric flux and these lines begin & terminate on +ve & -ve respectively but the magnetic flux lines are closed and do not terminate on a magnetic charge.

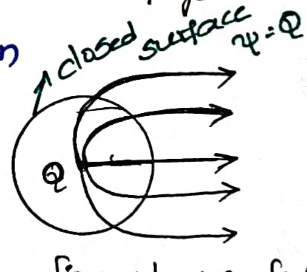


fig: Electric field lines

→ The magnetic flux lines are always closed so the net displacement is equal to zero $\boxed{\psi = 0}$ ***

(or)

***An [back side] isolated magnetic charge does not exist

$$\boxed{\psi = \oint_S \vec{B} \cdot d\vec{s} = 0}$$

This eqn is referred to as the Law of conservation of magnetic flux (or) Gauss Law for magnetostatic fields.

By applying divergence theorem

$$\oint_S \vec{B} \cdot d\vec{s} = \int_v (\nabla \cdot \vec{B})\, dv = 0 \Rightarrow \boxed{\nabla \cdot \vec{B} = 0}$$

→ This eqn is the Maxwell's 4th equation

Maxwell's equations for static EM fields :-

	Differential or Point form	Integral form	
1>	$\nabla\cdot\vec{D} = \rho_v$	$\oint_S \vec{D}\cdot\vec{ds} = \int_v \rho_v\, dv = Q$	Gauss Law [Maxwell – 4^{th} eqn]
2>	$\nabla\cdot\vec{B} = 0$	$\oint_S \vec{B}\cdot\vec{ds} = 0$	in Magnetostatic field
3>	$\nabla\times\vec{E} = 0$	$\oint_L \vec{E}\cdot\vec{dl} = 0$	electrostatic field [Maxwell – 2^{nd} eqn]
4>	$\nabla\times\vec{H} = \vec{J}$	$\oint_L \vec{H}\cdot\vec{dl} = \int_S \vec{J}\cdot\vec{ds}$	Amperes Law

Magnetic Scalar & Vector Potential :-

→ Just as $\vec{E} = -\nabla V$, the magnetic scalar potential V_m (in amp's) is defined as

$$\boxed{\vec{H} = -\nabla V_m \quad \text{if} \quad \vec{J} = 0} \quad — ①$$

W.K.T

$$\nabla\times\vec{H} = \vec{J}$$

$$\nabla\times -\nabla V_m = 0$$

$$\nabla\cdot\vec{B} = 0 \qquad \nabla\cdot\vec{H} = 0$$

$$\nabla\cdot\mu\vec{H} = 0 \qquad \nabla\cdot -\nabla V_m = 0$$

$$\boxed{\nabla^2 V_m = 0}$$

Laplace eqn in magnetic fields

→ Magnetic Vector Potential $\vec{A}$ (in wb/m) is defined as

$$\boxed{\vec{B} = \nabla\times\vec{A}}$$

derivation of Magnetic Vector Potential ($\vec{A}$)

consider Biot-Savart's Law

$$\vec{H} = \int_L \frac{I\,\vec{dl}\times\vec{R}}{4\pi R^3} \quad — ①$$

$$\vec{B} = \mu_0\vec{H}$$

$$\Rightarrow \vec{B} = \frac{\mu_0}{4\pi}\int_L \frac{I\,\vec{dl}\times\vec{R}}{R^3} \quad — ②$$

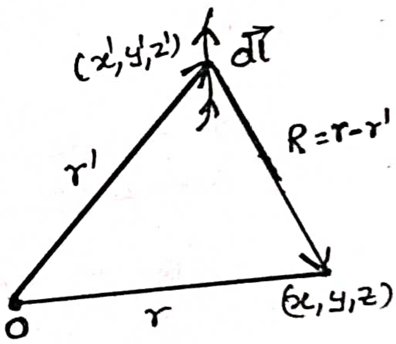


fig: illustration of the source point (x', y', z') and the field point (x, y, z)

$$\vec{R} = (x, y, z) \text{ to } (x', y', z')$$

$$= (x-x')\vec{a_x} + (y-y')\vec{a_y} + (z-z')\vec{a_z}$$

$$|R| = R = \left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{1/2}$$

$$\nabla(1/R) = \frac{d}{dx}\left(\frac{1}{R}\right)\vec{a_x} + \frac{d}{dy}\left(\frac{1}{R}\right)\vec{a_y} + \frac{d}{dz}\left(\frac{1}{R}\right)\vec{a_z}$$

$$\nabla(1/R) = \frac{-\vec{R}}{R^3}$$ — proof backside

$$\vec{B} = \frac{\mu_0}{4\pi}\int_L \frac{I\,\vec{dl}\times\vec{R}}{R^3} = \frac{-\mu_0}{4\pi}\int_L I\,\vec{dl}\times\nabla(1/R) \quad \text{— ③}$$

Now apply the vector identity

$$\nabla\times(f\vec{F}) = f\,\nabla\times\vec{F} + (\nabla f)\times\vec{F}$$

where f is a scalar field and $\vec{F}$ is a vector field. Taking

$f = \frac{1}{R}$ and $\vec{F} = \vec{dl}$

$$\nabla\times\left(\frac{1}{R}\vec{dl}\right) = \frac{1}{R}\nabla\times\vec{dl} + (\nabla(1/R))\times\vec{dl}$$

$\nabla A\times\vec{B} = -\vec{B}\times\nabla A$

$$\Rightarrow \vec{dl}\times\nabla(1/R) = \frac{1}{R}\nabla\times\vec{dl} - \nabla\times(1/R\,\vec{dl}) \quad \text{— ④}$$

Since ∇ operates w.r.to (x,y,z) while $\vec{dl}$ is a function of (x',y',z') so $\nabla\times\vec{dl}=0$

$$\frac{d}{dx}x' + \frac{d}{dy}y' + \frac{d}{dz}z' = 0$$

Eqn ④ is reduced to

$$\vec{dl}\times\nabla(1/R) = -\nabla\times\left(\frac{\vec{dl}}{R}\right) \quad \text{— ⑤}$$

Substitute eqn ⑤ in eqn ③

$$\vec{B} = -\frac{\mu_0}{4\pi}\int_L I\left(-\nabla\times\frac{\vec{dl}}{R}\right) = \frac{\mu_0 I}{4\pi R}\int\nabla\times\vec{dl}$$

$$= \nabla\times\int\frac{\mu_0 I\,\vec{dl}}{4\pi R} = \nabla\times\vec{A}$$

From that *** $\vec{A} = \int_L \frac{\mu_0 I\,\vec{dl}}{4\pi R}$ For line current

$$\vec{A} = \int_S \frac{\mu_0\vec{K}\,ds}{4\pi R}$$ For surface current

$$\vec{A} = \int_v \frac{\mu_0\vec{J}\,dv}{4\pi R}$$ For volume current

W.K.T

$$\psi = \int_S \vec{B}\cdot\vec{ds} = \int_S (\nabla\times\vec{A})\cdot\vec{ds}$$

Applying Stroke's theorem

$$\oint_L \vec{A}\cdot\vec{dL} = \int_S (\nabla\times\vec{A})\cdot\vec{ds} = \psi$$

$$\therefore \boxed{\psi = \oint_L \vec{A}\cdot\vec{dL}}$$

Vector identity

$$\nabla\times\nabla\times\vec{A} = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}$$

for a static magnetic field

$$\nabla\times\vec{B} = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}$$

& $\nabla\cdot\vec{A} = 0 \Rightarrow$

$$\nabla\times\vec{B} = -\nabla^2\vec{A}$$

$$\nabla\times\mu_0\vec{H} = -\nabla^2\vec{A}$$

$$\mu_0\,\nabla\times\vec{H} = -\nabla^2\vec{A} \rightarrow \nabla^2\vec{A} = -\mu_0\vec{J}$$

$\boxed{\nabla^2\vec{A} = -\mu_0\vec{J}}$ is called as Vector Poisson's eqn similar to

$\boxed{\nabla^2 V = -\rho_v/\varepsilon}$ in Electrostatic fields.

From Stroke's theorem

$$\oint_L \vec{H}\cdot\vec{dL} = \int_S (\nabla\times\vec{H})\cdot\vec{ds}$$

$\vec{B} = \mu_0\vec{H}$

$$= \frac{1}{\mu_0}\int_S (\nabla\times\vec{B})\cdot\vec{ds}$$

$$= \frac{1}{\mu_0}\int_S (-\nabla^2\vec{A})\cdot\vec{ds}$$

$$= \frac{+1}{\cancel{\mu_0}}\int_S +\cancel{\mu_0}\vec{J}\cdot\vec{ds}$$

$$\boxed{\oint_L \vec{H}\cdot\vec{dL} = \int_S \vec{J}\cdot\vec{ds} = I_{enc}}$$

which is Amperes circuit Law

Inductor and Inductance :-

An insulated conducting wire wound on a magnetic material with N turns called a coil (or) inductor. The medium inside the coil may be a magnetic material (or) freespace.

When a current I passes through the coil, it produces a magnetic field 'B' which causes a flux ψ in each turn of the coil. Inductors store energy in magnetic fields.

If the ckt has 'N' identical turns, the flux linkage 'λ' is defined as $\boxed{\lambda = N\psi}$ product of N & ψ

Where N is the no. of turns in a coil &
ψ is the magnetic flux passing through it

* flux linkage is the product of no. of turns in a coil and the magnetic flux passing through it.

If the medium of the coil is linear, then the flux linkage of the coil is proportional to the current flowing through it.

$\underbrace{N\psi}_{\lambda} \propto I$ (or) $\boxed{\lambda = LI}$

Where L is a proportionality constant (or) constant of proportionality called the Inductance of the circuit.

$\boxed{L = \frac{N\psi}{I}}$ Wb/Amp (or) Henry (or) $L = \frac{\lambda}{I}$

The ratio of the total flux linkage to the current flowing through the coil is ~~(defined)~~ called as Inductance (L)

* This is also called as self inductance of the coil since the flux is produced by the coil itself.

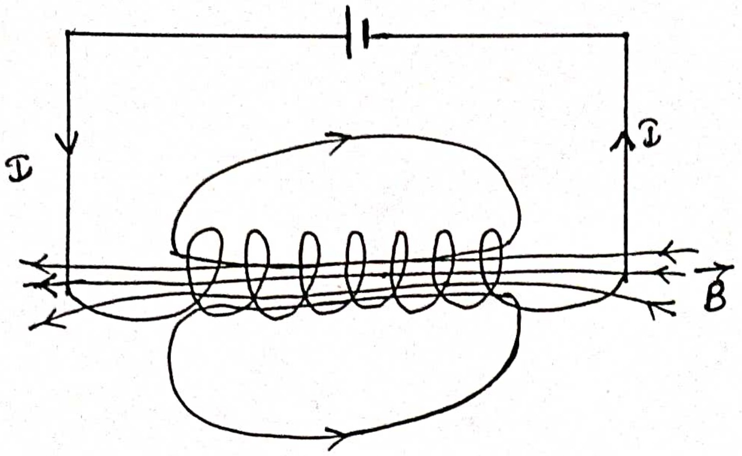


fig: Magnetic field $\vec{B}$ produced by a circuit.

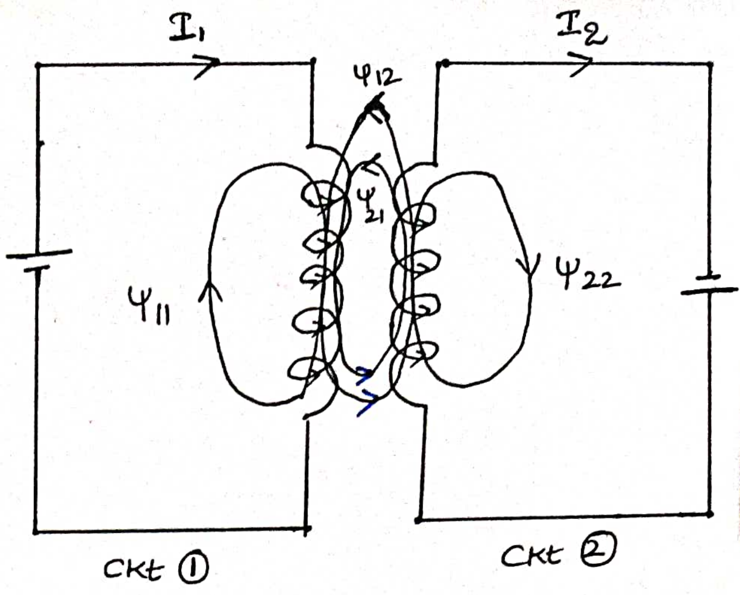


fig: Magnetic interaction between two circuits.

If there are two ckts carrying currents I_1 and I_2 as shown in fig: A magnetic interaction exists between them. Four component fluxes ψ_{11}, ψ_{12}, ψ_{21} and ψ_{22} are produced.

ψ_{12} represents the flux passing through ckt ① due to current I_2 in ckt ②. If $\vec{B_2}$ is the field due to I_2 and S_1 is the area of ckt ① then

$$\psi_{12} = \int_{S_1} \vec{B_2} \cdot \vec{ds}$$

The inductance due to the flux linkages between the coils is called mutual inductance

<u>Mutual inductance</u> M_{12} is defined as the ratio of the flux linkage on ckt ① to current I_2 in ckt ②

Flux linkage in ckt ① due to current I_2 in ckt ②

$$\lambda_{12} = N_1 \psi_{12}$$

$$M_{12} = \frac{\lambda_{12}}{I_2} = \frac{N_1 \psi_{12}}{I_2} \text{ Henrys} \qquad \text{III}^{ly} \quad M_{21} = \frac{\lambda_{21}}{I_1} = \frac{N_2 \psi_{21}}{I_1}$$

Note: If the medium surrounding the ckts is linear in the absence of ferromagnetic material $\boxed{M_{12} = M_{21}}$

The <u>Self Inductance</u> of ckts ① & ② respectively is defined as

$$L_1 = \frac{\lambda_{11}}{I_1} = \frac{N_1 \psi_1}{I_1} \; ; \quad L_2 = \frac{\lambda_{22}}{I_2} = \frac{N_2 \psi_2}{I_2}$$

where $\psi_1 = \psi_{11} + \psi_{12}$ and $\psi_2 = \psi_{21} + \psi_{22}$

The total energy in the magnetic field is the sum of the Energies due to L_1, L_2 and M_{12} (or M_{21})

$$W_m = W_1 + W_2 + W_{12}$$
$$= \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 \pm \frac{1}{2} M_{12} I_1 I_2$$

+ve sign taken if I_1 and I_2 flow such that the magnetic fields of two circuits strengthen each other (aiding)

−ve sign taken if the currents flow such that the magnetic fields of two circuits oppose each other

Magnetic energy :-

Consider a differential volume in a magnetic field as shown in fig: Let the volume be covered with conducting sheets at the top & bottom surfaces with current ΔI

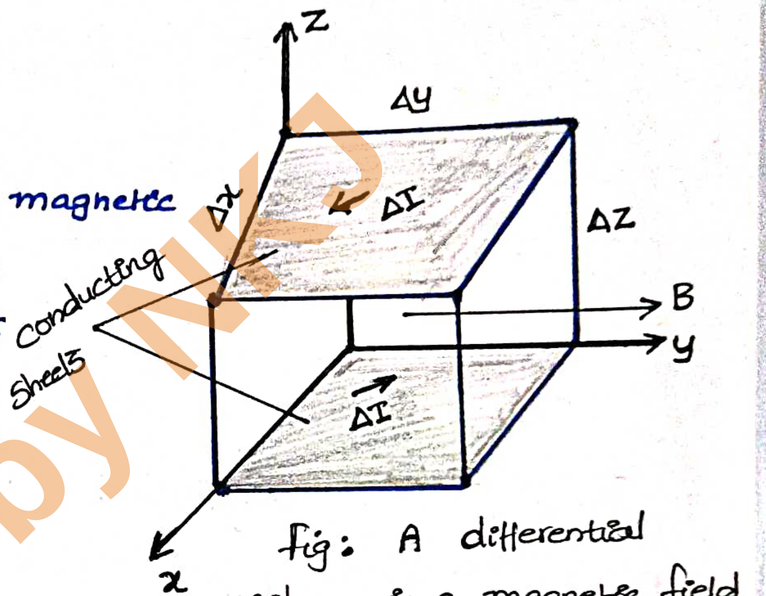


Fig: A differential volume in a magnetic field

W.K.T $L = \frac{N\psi}{I}$ for one turn $N=1$

$L = \frac{\psi}{I} \Rightarrow \Delta L = \frac{\Delta\psi}{\Delta I}$

$\psi = \int B \cdot \vec{ds}$ along y-direction $\psi = \Delta y \cdot B$

Assume the whole region is filled with such differential volumes. each volume has an inductance

that means the flux passes through each volume

$$\Delta L = \frac{\Delta\psi}{\Delta I} = \frac{\Delta x \Delta y \Delta z}{\Delta I} = \frac{B\,\Delta x \Delta z}{\Delta I} = \frac{\mu H \Delta x \Delta z}{\Delta I}$$

$\psi = Wb/m^2$

from ohms' Law $H = \frac{I}{l} \Rightarrow H = \frac{\Delta I}{\Delta l} \Rightarrow \Delta I = H \times \Delta l$

↳ Static Magnetic field

Substitute in the above eqn $W = \frac{1}{2} L I^2$

$\vec{B}$ is along the y-direct $\Delta y = \vec{B}$

$$\Delta w = \frac{1}{2}\Delta L \Delta I^2 = \frac{1}{2} \Delta x \Delta y \Delta l \times \mu H^2 = \frac{1}{2}\Delta x \Delta y \Delta z = \frac{1}{2} \Delta v \times \mu H^2$$

Δl along y-direct $= \Delta y$

$$w = \int \frac{1}{2} \mu H^2 dv$$

Magnetic energy density $w_m\ (J/m^3) = \frac{d}{dv}(w) = \boxed{\frac{1}{2}\mu H^2}$
$= \boxed{\frac{1}{2} \vec{B} \cdot \vec{H}}$

$$\therefore \quad P = j\omega\sqrt{LC}\left[1 + \frac{1}{2}\frac{R}{j\omega L} + \cdots\right]\left[1 + \frac{1}{2}\frac{G}{j\omega C} + \cdots\right]$$

neglecting the higher order terms ($R/\omega L \ll 1$ & $G/\omega C \ll 1$)

$$P = j\omega\sqrt{LC}\left[1 + \frac{1}{2}\frac{R}{j\omega L} + \frac{1}{2}\frac{G}{j\omega C} + \cdots\right] \qquad \frac{1}{4}\frac{RG}{j\omega LC} \ll 1$$

$$P = j\omega\sqrt{LC}\left[1 + \frac{1}{2}\frac{R}{j\omega L} + \frac{1}{2}\frac{G}{j\omega C}\right]$$

$$= \left[j\omega\sqrt{LC} + \frac{1}{2}j\omega\sqrt{LC} \times \frac{R}{j\omega L} + j\omega\sqrt{LC} \times \frac{1}{2}\frac{G}{j\omega C}\right]$$

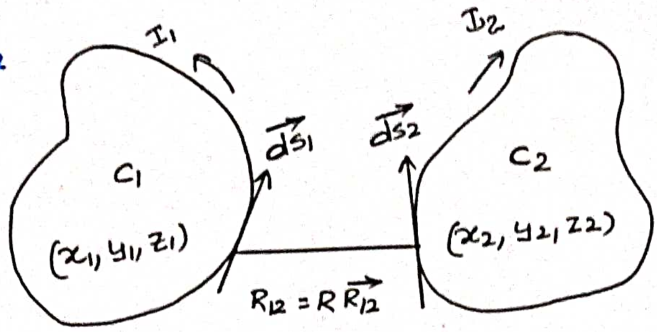
$$P = \alpha + j\beta = \left[j\omega\sqrt{LC} + \frac{R}{2}\sqrt{\frac{C}{L}} + \frac{G}{2}\sqrt{\frac{L}{C}}\right]$$

Equating real and imaginary terms

$$\boxed{\alpha = \frac{R}{2}\sqrt{\frac{C}{L}} + \frac{G}{2}\sqrt{\frac{L}{C}}} \qquad \alpha = \frac{R}{2}\sqrt{\frac{C}{L}} + \frac{G}{2}\sqrt{\frac{L}{C}}$$

$$\boxed{\beta = \omega\sqrt{LC}} \; ; \qquad V_p = \cdots = \frac{1}{\sqrt{LC}}$$

Ampere's Force Law :-



Consider two Loops designated C_1 & C_2 carrying currents I_1 & I_2. The quantities $\vec{ds_1}$ and $\vec{ds_2}$ are directed (differential) elemental lengths and the directed distance between them is $R\vec{R_{12}}$.

Where $\vec{R_{12}}$ = unit vector drawn from point ① to point ② Then Ampere's Force Law is

$$\vec{F_{21}} = \frac{\mu}{4\pi} I_1 I_2 \oint_{C1} \oint_{C2} \frac{\vec{ds_2} \times (\vec{ds_1} \times \vec{R_{12}})}{R^2}$$

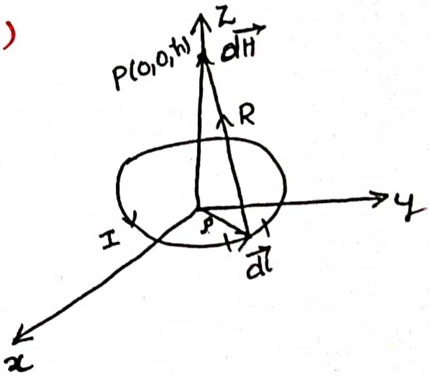
$\vec{F_{21}}$ is the Force on ckt C_2 due to ckt C_1

W.K.T $\vec{H_{21}} = \frac{I_1}{4\pi} \oint_{C_1} \frac{\vec{ds_1} \times \vec{R_{12}}}{R^2}$ $\left[\cancel{\text{and } \vec{H_{21}} = \frac{I_1}{4\pi}}\right]^{\times}$; $\vec{H} = \frac{I}{4\pi} \int \frac{\vec{dl} \times \vec{a_R}}{R^2}$

then $\vec{F_{21}} = \mu I_2 \oint_{C2} \vec{ds_2} \times \vec{H_{21}} = \underline{I_2 \oint_{C2} \vec{ds_2} \times \vec{B_{21}}}$

Problems :-

① A circular Loop Located on $x^2+y^2=9$, $z=0$ carries a direct current of 10A along a_ϕ. determine $\vec{H}$ at $(0,0,4)$ and $(0,0,-4)$



Sol: $\vec{dH} = \frac{I\,\vec{dl} \times \vec{R}}{4\pi R^3}$

$\vec{R} = (0,0,h) - (x,y,0)$

$= -x\vec{a_x} - y\vec{a_y} + h\vec{a_z}$

Substitute $\vec{a_x} = \cos\phi\,\vec{a_\rho} - \sin\phi\,\vec{a_\phi}$

$\vec{a_y} = \sin\phi\,\vec{a_\rho} + \cos\phi\,\vec{a_\phi}$

$x = \rho\cos\phi$ & $y = \rho\sin\phi$ then $\vec{R} = -\rho\vec{a_\rho} + h\vec{a_z}$; $|R| = R = (\rho^2+h^2)^{1/2}$

$\vec{dl}$ along $\vec{a_\phi}$ $= \rho\, d\phi\, a_\phi$

$$\vec{dl} \times \vec{R} = \begin{vmatrix} \vec{a_\rho} & \vec{a_\phi} & \vec{a_z} \\ 0 & \rho d\phi & 0 \\ -\rho & 0 & h \end{vmatrix} = \rho h\, d\phi\, \vec{a_\rho} + \rho^2 d\phi\, \vec{a_z}$$

$$\vec{dH} = \frac{I\left(\rho h\, d\phi\, \vec{a_\rho} + \rho^2 d\phi\, \vec{a_z}\right)}{4\pi(\rho^2+h^2)^{3/2}}$$

$\vec{a_\rho} = \cos\phi\,\vec{a_x} + \sin\phi\,\vec{a_y}$; Integrating $\cos\phi$ (or) $\sin\phi$ over $0 \le \phi \le 2\pi$ gives zero, the first term is equal to zero. then

$$\vec{H} = \int_0^{2\pi} \frac{I\rho^2 d\phi\, \vec{a_z}}{4\pi(\rho^2+h^2)^{3/2}} = \frac{I\rho^2 \vec{a_z}}{2(\rho^2+h^2)^{3/2}}$$

$(0,0,h)$

(a) Substitute $I = 10A$, $\rho = 3$, $h = 4$ gives

$x^2+y^2 = 9 = \rho^2$
$\rho = 3$

$$H(0,0,4) = \frac{10 \times 9 \times \vec{a_z}}{2(9+16)^{3/2}} = 0.36\,\vec{a_z} \text{ A/m}$$

(b) $H(0,0,-4) = 0.36\,\vec{a_z}$ A/m.

② Plane $y=1$ carries current $K = 50\,\vec{a_z}$ mA/m. Find $\vec{H}$ at ⓐ $(0,0,0)$ ⓑ $(1,5,-3)$

Sol: $\vec{H} = \frac{1}{2}\vec{K} \times \vec{a_n}$

$$\vec{H}(0,0,0) = +\frac{1}{2} \times 50\,\vec{a_z} \times \vec{a_y} \text{ mA/m} \quad z<0 \rightarrow H =$$

$$= -25\,\vec{a_x} \text{ mA/m}$$

$$\vec{H}(1,5,-3) = \frac{1}{2} \times 50\,\vec{a_z} \times \vec{a_y} \quad z>0$$

$$= -25\,\vec{a_x} \text{ mA/m}$$

③ In a conducting medium $\vec{H} = y^2 z\,\vec{a_x} + 2(x+1)yz\,\vec{a_y} - (x+1)z^2\,\vec{a_z}$ A/m find the current density at $(1,0,-3)$

Sol: $$\nabla \times \vec{H} = \vec{J} = \begin{vmatrix} \vec{a_x} & \vec{a_y} & \vec{a_z} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ H_x & H_y & H_z \end{vmatrix} = 9\,\vec{a_y} \text{ A/m}^2$$

④ A current distribution gives rise to the vector magnetic potential $\vec{A} = x^2y\,\vec{a_x} + y^2x\,\vec{a_y} - 4xyz\,\vec{a_z}$ Wb/m. Calculate ⓐ $\vec{B}$ at $(-1,2,5)$ ⓑ The flux through the surface is defined by $z=1$, $-1 \le y \le 4$, $0 \le x \le 1$.

Sol: $$\vec{B} = \nabla \times \vec{A} = \begin{vmatrix} \vec{a_x} & \vec{a_y} & \vec{a_z} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ x^2y & y^2x & -4xyz \end{vmatrix} = -4xz\,\vec{a_x} + 4yz\,\vec{a_y} + (y^2-x^2)\,\vec{a_z}$$

$$\vec{B}_{at\,(-1,2,5)} = 20\,\vec{a_x} + 40\,\vec{a_y} + 3\,\vec{a_z}$$

$$\psi = \oint \vec{B} \cdot \vec{ds} = \int \left((-4xz)\vec{a_x} + 4yz\,\vec{a_y} + (y^2-x^2)\vec{a_z}\right)(dydz\,\vec{a_x} + dxdz\,\vec{a_y} + dxdy\,\vec{a_z})$$

$$= \int (-4xz)dydz + \int 4yz\,dxdz + \int (y^2-x^2)dxdy$$

$z=1$
$dz=0$

$$= \int_0^1 \int_{-1}^4 (y^2-x^2)\,dxdy = \boxed{20 \text{ wb}}$$

FORCE DUE TO MAGNETIC FIELDS:

Backside.

[Lorentz force equation]

Consider a +ve charge q placed in static electric & static magnetic fields.

A. Force on a charged particle :-

The electric field exerts a force on stationary as well as moving charges. From coulomb's law the force acting on a charged particle is given by

$$\boxed{F_e = qE} \quad \text{--- (1)}$$

F_e = electric force

The direction of force is same as that of the electric field intensity. No magnetic force is exerted on a stationary charge.

But a Magnetic field can exert force only on a moving charge. The Magnetic force F_m experienced by a charge 'q' moving with a velocity 'u' in a magnetic field 'B' is

$$\boxed{F_m = q\,u \times B} \quad \text{--- (2)}$$

F_m is ⊥ar to both V & B.

For a moving charge "q" in the presence of both electric & magnetic fields, the total force on the charge is given by

$$\boxed{\begin{array}{l} F = F_e + F_m \\ \hline F = q(E + u \times B) \end{array}} \quad \text{--- (3)}$$

This is known as the Lorentz force eqn. It relates mechanical force to electrical force.

Lorentz force Law is a fundamental Law that gives the relation between the forces on the particle due to the combined impact of electric & magnetic fields.

If the mass of the charged particle moving in E & B fields is "m". by Newton's 2nd Law of motion.

$$\boxed{F = ma = m\frac{dv}{dt} = q(E + u \times B)} \quad \text{--- (4)}$$

B. FORCE ON A CURRENT ELEMENT :

Consider the convection current density

$$\boxed{J = \rho_v u} \quad — \quad ④$$

W.K.T

The relationship between current elements

$$\underline{I dl} = K ds = \underline{J dv} \rightarrow \text{eqn } ①$$

$$\rho_v = \frac{dq}{dv}$$

$$\boxed{I dl = \rho_v u \, dv = dq \, u} \quad — \quad ② + ③ - ⑤$$

(or) alternatively

$$I = \frac{dq}{dt} \rightarrow I dl = \frac{dq}{dt} \times dl$$

$$= dq \cdot \frac{dl}{dt} \qquad \boxed{u = \frac{d}{t}}$$

$$= dq \cdot u$$

Hence $\boxed{I dl = dq \, u}$ — ⑥

The above eqn shows that an elemental charge dq moving with velocity 'u' is equal to a conduction current element Idl.

from eqn ⑨

$$F_m = q \, u \times B$$

The differential magnetic force exerted on the differential charge is

$$\underline{dF_m} = \underline{dq} \, u \times B$$

$$\boxed{dF_m = I dl \times B} \quad — \quad ⑦$$

from eqn ⑦ The magnetic field 'B' is defined as the force per unit current element.

If the current 'I' is through a closed path L (or) ckt the force on the circuit is given by

$$\boxed{F = \oint_L I dl \times B} \quad — \quad ⑧ \quad - \text{for Line current}$$

$|||^{ly}$ $dF_m = K\,ds \times B$ $dF_m = J\,dv \times B$

For surface current $F = \int_S K\,ds \times B$ $F = \int_v J\,dv \times B$ For volume current.

C. FORCE BETWEEN TWO CURRENT ELEMENTS (OR) AMPERE'S FORCE LAW

States that the force between two current carrying elements $I_1 dl_1$ and $I_2 dl_2$ placed at a distance R_{12} is given by

$$\vec{F} = \frac{\mu}{4\pi} \oint_{L_1} \oint_{L_2} \frac{I_2 \vec{dl_2} \times (I_1 \vec{dl_1} \times \vec{a_{R12}})}{R_{12}^2}$$

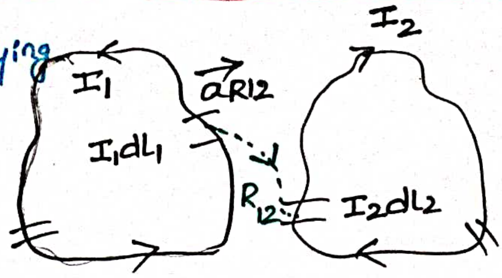


fig: Force between two current Loops.

This Law is used to express the force on one current carrying conductor directly interms of 2nd current carrying conductor without needing to find the magnetic field between them.

Proof : Consider two conductors in the form of Loop L_1 and Loop L_2 carrying currents I_1 & I_2 respectively as shown in fig:

Let the current element of the first conductor be $I_1 dL_1$ and that of the 2nd conductor be $I_2 dl_2$. The current elements are placed at a distance R_{12} from each other, the unit vector $\vec{a_{R12}}$ is as shown in fig:

According to Biot-Savart's Law, the magnetic field intensity at $I_2\vec{dl_2}$ (Loop 2) due to current element $I_1\vec{dl_1}$ is

$$dH = \frac{I\,dL \times \vec{a_R}}{4\pi R^2}$$

$$dH_2 = \frac{I_1 \vec{dl_1} \times \vec{a_{R12}}}{4\pi R_{12}^2}$$

$$B_2 = \mu_0 H_2$$

(or) $$dB_2 = \frac{\mu_0 I_1 \vec{dl_1} \times \vec{a_{R12}}}{4\pi R_{12}^2}$$

Due to the magnetic field B_2, the differential force on the current element $I_2 \vec{dl_2}$ is

from eqn (7) $$dF_2 = I_2 \vec{dl_2} \times B_2$$

Taking differential magnetic flux density

$$d(dF_2) = I_2 \vec{dl_2} \times dB_2$$

$$d(dF_2) = \frac{I_2 \vec{dl_2} \times \mu_0 I_1 \vec{dl_1} \times \vec{a_{R12}}}{4\pi R_{12}^2} \Rightarrow \vec{F_2} = \left[\oint_{L_1}\oint_{L_2} \frac{I_2 \vec{dl_2} \times I_1(\vec{dl_1} \times \vec{a_{R12}})}{R_{12}^2}\right]\frac{\mu_0}{4\pi}$$

Note $F_2 = -F_1$ both are equal but opposite in sign.

The Variation of flux with time can be in three Ways

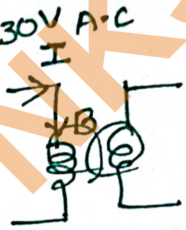
a) By having a stationary Loop in a time varying $\vec{B}$ field

b) " " " time varying " " " static $\vec{B}$ field

c) " " " " " " time varying $\vec{B}$ field

When an emf is induced in a stationary Loop due to time Varying magnetic field the e.m.f is called as statically induced e.m.f (or) transformer e.m.f.

When an emf is induced in a time varying Loop due to Static magnetic field $\vec{B}$ then the e.m.f is called dynamically induced e.m.f (or) motional e.m.f.



Case A : Transformer e.m.f

In a transformer coil is stationary due to alternating current some magnetic field is produced. This magnetic field again Produces some alternating current inside the coil. In case A also Loop is stationary $\frac{d}{dt}$ operates only on B(t) that why it is called as Transformer e.m.f

Case B : Motional e.m.f it is due to the motion of a conductor in a magnetic field.

Inconsistency of Ampere's Law: $\nabla \cdot \vec{J} = -\frac{d\rho_v}{dt}$ so charges are constant for static Em fields so ρ_v constant $\frac{d}{dt}(\rho_v) = \frac{d}{dt}(\text{constant}) = 0$ But for time varying fields ρ_v is not constant. it is varying (or) changing with time. So Amperes Law is not applicable for time varying fields. So to overcome the problem of inconsistency of A.L $\nabla \times \vec{H} = \vec{J}$ is modified as $\nabla \times \vec{H} = \vec{J} + \vec{J_d}$.

UNIT- 3

Maxwell's Equations (time varying fields):

→ Electrostatic fields are usually produced by static electric charges where Magnetostatic fields are due to motion of electric charges with uniform velocity (direct current); time varying fields or waves are usually due to accelerated charges or time varying currents.

→ static $\vec{E}$, magnetic fields does not vary with time, where as electromagnetic fields (waves) are varying or changing with time.

FARADAY's Law :-

According to Faraday's experiment, a static magnetic field produces no current flow but a time varying fields produces an induced voltage (called electromotive force or simply emf) in a closed circuit, which causes a flow of current.

Faraday discovered that the induced emf, V_{emf} (in volts) in any closed circuit is equal to the time rate of change of the magnetic flux linkage by the circuit.

This is called Faraday's law, and it can be expressed as

$$\boxed{V_{emf} = \frac{-d\lambda}{dt} = -N\frac{d\psi}{dt}} \qquad (\lambda = N\psi)$$

where N is the number of turns in the circuit and ψ is the flux through each turn. The negetive sign shows that the induced voltage acts in such a way as to oppose the flux producing it. This is known as Lenz's law and it emphasizes the fact that the direction of current flow in the circuit is such that the induced magnetic field produced by the induced current will oppose the original magnetic field.

Transformer and motional EMFs :

According to Faraday's law induced voltage $V_{emf} = -\frac{d\psi}{dt}$ — ① for single turn $(N=1)$

In terms of $\vec{E}$ and $\vec{B}$, eqn ① can be written as

$$V_{emf} = +\oint_L \vec{E}\cdot\vec{dl} = -\frac{d}{dt}\int_S \vec{B}\cdot\vec{ds} \qquad \psi = \int_S \vec{B}\cdot\vec{ds}$$

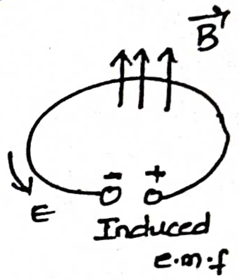
where 'S' is the surface area of the circuit bounded by the closed path 'L'.

case A : Stationary Loop in Time-Varying $\vec{B}$ field (Transformer emf)

Here a stationary conducting loop is in a time varying magnetic $\vec{B}$ field.

$$V_{emf} = \oint_L \vec{E}\cdot\vec{dl} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot\vec{ds}$$

Since $\vec{B}$ may change with co-ordinates as well as time so partial derivative is considered



Apply a Strokes theorem

$$\oint_L \vec{E}\cdot\vec{dl} = \iint_S (\nabla\times\vec{E})\cdot\vec{ds} = -\iint_S \frac{\partial\vec{B}}{\partial t}\cdot\vec{ds}$$

Electromotive force ← $\boxed{\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}}$

This emf induced by the time-varying current (Producing the time varying $\vec{B}$ field) in a stationary loop is referred to as transformer emf.

case B : Moving Loop in static $\vec{B}$ field (Motional emf)

when a conducting Loop is moving in a static $\vec{B}$ field, an emf is induced in the Loop. The force on a charge moving with uniform velocity $\vec{v}$ in a magnetic field $\vec{B}$ is $F_m = Q\vec{v}\times\vec{B}$

Motivational electric field is defined as $(E_m) \Rightarrow \vec{E}_m = \frac{\vec{F}_m}{Q}$

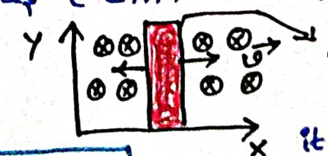


fig: Closed path moving in static $\vec{B}$ field with Velocity $\vec{v}$

$$\vec{E}_m = \frac{\vec{F}_m}{Q} = \vec{v}\times\vec{B}$$

It is due to motional action

Now $\boxed{V_{emf} = \oint_L \vec{E}\cdot\vec{dl} = \oint_L (\vec{v}\times\vec{B})\cdot\vec{dl}}$

This type of emf is called motional emf or flux cutting emf beca

Case c: Moving Loop in time-varying field :-

This is the general case in which a moving conducting loop is in a time-varying magnetic field. Both transformer emf and motional emf are present. Total induced e.m.f = Transformer e.m.f + motional e.m.f.

$$V_{emf} = \oint_L \vec{E}\cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s} + \oint_L (\vec{u}\times\vec{B})\cdot d\vec{l}$$

Apply a Stoke's theorem.

$$\oint_L \vec{E}\cdot d\vec{l} = \int_S (\nabla\times\vec{E})\cdot d\vec{s} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s} + \oint_L (\vec{u}\times\vec{B})\cdot d\vec{l}$$

for seperate term

$$\oint_L (\vec{u}\times\vec{B})\cdot d\vec{l} = \int_S (\nabla\times(\vec{u}\times\vec{B}))\cdot d\vec{s}$$

then

$$\int_S (\nabla\times\vec{E})\cdot d\vec{s} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s} + \int_S \nabla\times(\vec{u}\times\vec{B})\cdot d\vec{s}$$

$$\boxed{\nabla\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} + \nabla\times(\vec{u}\times\vec{B})}$$

Displacement Current (or) Inconsistency of Ampere's law :

For static EM fields $\nabla\times\vec{H} = \vec{J}$

But the divergence of the curl of any vector field is identically zero

Proof backside $\leftarrow$ $\nabla\cdot(\nabla\times\vec{H}) = \vec{0} = 0 = \nabla\cdot\vec{J}$ — ①

But the continuity of current eqn $\nabla\cdot\vec{J} = -\frac{\partial \rho_v}{\partial t} \neq 0$ — ②

Thus the two equations $\nabla\cdot\vec{J} = 0$ & $\nabla\cdot\vec{J} = -\frac{\partial \rho_v}{\partial t}$ are not compatible for time varying fields on conditions. To agree with eqn ② the eqn $\nabla\times\vec{H} = \vec{J}$ is modified as $\nabla\times\vec{H} = \vec{J} + \vec{J_d}$

$$\nabla\cdot(\nabla\times\vec{H}) = 0 \Rightarrow \nabla\cdot(\vec{J} + \vec{J_d}) = \nabla\cdot\vec{J} + \nabla\cdot\vec{J_d}$$

$$\nabla\cdot\vec{J} + \nabla\cdot\vec{J_d} = 0 \ ; \quad \nabla\cdot\vec{J_d} = -\nabla\cdot\vec{J} = \frac{\partial \rho_v}{\partial t}$$

$\nabla\cdot\vec{D} = \rho_v$

$$\nabla\cdot\vec{J_d} = \frac{\partial \rho_v}{\partial t} = \frac{d}{dt}(\nabla\cdot\vec{D}) = \nabla\cdot\frac{\partial \vec{D}}{\partial t}$$

$$*** \quad \boxed{\vec{J_d} = \frac{\partial \vec{D}}{\partial t}}$$

Substitute in $\nabla \times \vec{H} = \vec{J} + \vec{J}_d$

magnetomotive force ← $\boxed{\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}}$

$\nabla \times \vec{H} = \vec{J}$
↓
Ampere's ckt Law

This is the Maxwell's equation (based on Ampere's circuit Law) for a time-varying field. The term $\vec{J}_d = \frac{\partial \vec{D}}{\partial t}$ is known as displacement current density and $\vec{J}$ is the conduction current density ($J = \sigma E$)

Maxwell's Equations :-

Maxwell's equations are a set of four equations are derived from Ampere's circuit law

1> $$\oint_L \vec{H}\cdot \vec{dl} = I_{enc} = \int_S \vec{J}\cdot\vec{ds}$$

$$\oint_L \vec{H}\cdot \vec{dl} = \int_S (\vec{J}+\vec{J}_d)\cdot \vec{ds}$$

From Strokes theorem $$= \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right)\cdot\vec{ds}$$

$$\oint_L \vec{H}\cdot \vec{dl} = \int_S (\nabla\times\vec{H})\cdot\vec{ds} = \int_S \left(\vec{J}+\frac{\partial \vec{D}}{\partial t}\right)\cdot\vec{ds}$$

$$\boxed{\nabla\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}}$$

2>
From Faraday's Law

$$\oint_L \vec{E}\cdot\vec{dl} = -\int \frac{\partial \vec{B}}{\partial t}\cdot\vec{ds}$$

from Strokes theorem

2> $$\oint_L \vec{E}\cdot\vec{dl} = \int_S (\nabla\times\vec{E})\cdot\vec{ds} = -\int \frac{\partial \vec{B}}{\partial t}\cdot\vec{ds}$$

$$\boxed{\nabla\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

3>
From Gauss Law for Electrostatics $\nabla\cdot\vec{D} = \rho_v$

W.K.T $$\int_S \vec{D}\cdot\vec{ds} = \psi = Q \Rightarrow \int_v (\nabla\cdot\vec{D})\,dv$$ ← from divergence theorem

$$\int_S \vec{D}\cdot\vec{ds} = \int_v (\nabla\cdot\vec{D})\,dv = Q = \int_v \rho_v\,dv$$

$$\boxed{\nabla\cdot\vec{D} = \rho_v}$$

4> From Gauss law of magnetostatics

$$\psi = \int_S \vec{B}\cdot d\vec{s} = 0$$

$$\int_S \vec{B}\cdot d\vec{s} = \int_v (\nabla\cdot\vec{B})\, dv = 0$$ Leakage

$$\boxed{\nabla\cdot\vec{B} = 0}$$

Generalised forms of Maxwell's Equations:

Differential Form	Integral Form	Remarks
$\nabla\cdot\vec{D} = \rho_v$	$\oint_S \vec{D}\cdot d\vec{s} = \int_v \rho_v\, dv$	Gauss Law
$\nabla\cdot\vec{B} = 0$	$\oint_S \vec{B}\cdot d\vec{s} = 0$	Nonexistence of isolated magnetic charge
$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$	$\oint_L \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{s}$	Faraday's law
$\nabla\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}$	$\oint_L \vec{H}\cdot d\vec{l} = \int_S \left(\vec{J} + \frac{\partial\vec{D}}{\partial t}\right)\cdot d\vec{s}$	Amperes circuit Law.

Word Statements of the Field Equations :-

1> $\oint_S \vec{D}\cdot d\vec{s} = \int_v \rho_v\, dv$

The total electric displacement through the surface enclosing a volume is equal to the total charge with in the volume $\psi = \oint_S \vec{D}\cdot d\vec{s}$
$Q = \int_v \rho_v\, dv$

2> $\oint_S \vec{B}\cdot d\vec{s} = 0 = \psi$

The net magnetic flux emerging through any closed surface is zero.

$\oint \vec{E}\cdot d\vec{l} = V_{emf} = EMF$

3> $\oint_L \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{s}$.

The electromotive force around a closed path is equal to time derivative of the magnetic displacement through any surface bounded by the path.

4> $\oint_L \vec{H}\cdot d\vec{l} = \int_S \left(\vec{J} + \frac{\partial\vec{D}}{\partial t}\right)\cdot d\vec{s}$

The magnetomotive force around a closed path is equal to the conduction current plus the time derivative of the electric displacement through any Surface bounded by the path.

The 3rd & 4th Maxwell's can then be stated :-

1> Let $\oint_L \vec{H}\cdot d\vec{l}$ length $\int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right)\cdot d\vec{s}$ = Primary constants/km Assuming that these do not vary with freq.

short section of magnetic voltage around a closed path is equal to " From the sending end 'A'.

" the electric current through the path.

2> $\oint_L \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{s}$ magnetic current varies w.r.to time.

The electric voltage around a closed path is equal to the magnetic current through the path.

through

UNIT - III & IV

MAXWELL'S EQUATIONS

UNIT – 4

References

1. Jordan & Balmain

slide No:1

Maxwell's equations are used to determine the boundary conditions.

$$\oint \vec{E}\cdot\vec{dl} = 0 \quad \& \quad —①$$

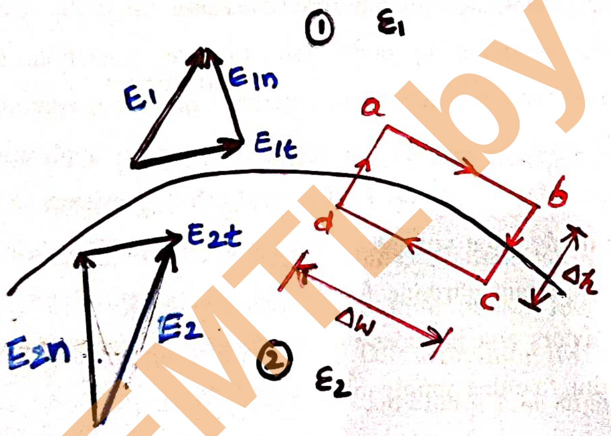
$$\oint \vec{D}\cdot\vec{ds} = Q_{enc} = \psi \quad —②$$

Electric field intensity $\vec{E}$ is divided in to two orthogonal components $\vec{E} = \vec{E_t} + \vec{E_n}$

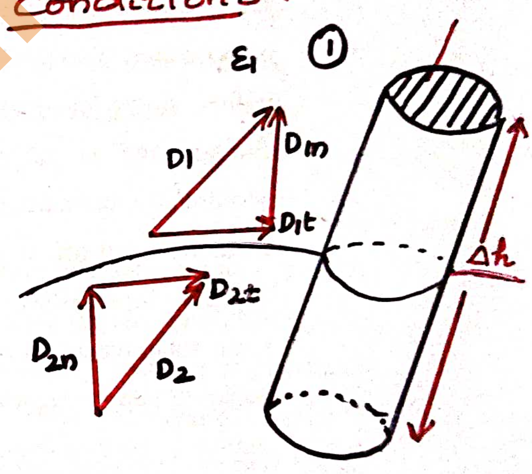
$\vec{E_t} \rightarrow$ tangential Component of $\vec{E}$

$\vec{E_n} \rightarrow$ Normal component of $\vec{E}$

(A) Dielectric – Dielectric Boundary conditions:



(a)



(b)

fig : Dielectric – dielectric boundary

$$0 = \underbrace{E_{1t}\Delta w}_{a \text{ to } b} - \underbrace{E_{1n}\frac{\Delta h}{2}}_{b \text{ to Boundary}} - \underbrace{E_{2n}\frac{\Delta h}{2}}_{B \text{ to } c} - \underbrace{E_{2t}\Delta w}_{c \text{ to } d} + \underbrace{E_{2n}\frac{\Delta h}{2}}_{d \text{ to } B} + \underbrace{E_{1n}\frac{\Delta h}{2}}_{B \text{ to } a}.$$

As $\Delta h \rightarrow 0$

$$E_{1t}\Delta w = E_{2t}\Delta w$$

$$\boxed{E_{1t} = E_{2t}} \quad — ①$$

Slide no: 2

consider the $\vec{E}$ field existing in a region consisting of two different dielectrics characterized by $\varepsilon_1 = \varepsilon_0 \varepsilon_{r1}$ and $\varepsilon_2 = \varepsilon_0 \varepsilon_{r2}$ as shown in fig:

$\vec{E_1}$ is the electric field in media ①
$\vec{E_2}$ is the electric field in media ②

$\vec{E_1}$ & $\vec{E_2}$ can be decomposed as

$$\vec{E_1} = \vec{E_{1t}} + \vec{E_{1n}}$$

$$\vec{E_2} = \vec{E_{2t}} + \vec{E_{2n}}$$

Now apply a closed path abcda of fig (a) assuming that the path is very small w.r.to the variation of $\vec{E}$.

$$\oint \vec{E} \cdot \vec{dl} = 0$$

from eqn ①

Thus the tangential components of $\vec{E}$ are the same on the two sides of the boundary.

In other words E_t undergoes no change on the boundary and it is said to be continuous across the boundary

$$\vec{D} = \varepsilon \vec{E} \Rightarrow \vec{E} = \frac{\vec{D}}{\varepsilon}$$

$$\frac{D_{1t}}{\varepsilon_1} = E_{1t} = E_{2t} = \frac{D_{2t}}{\varepsilon_2}$$

$$\boxed{\frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}}$$

$D_{1t} = D_{2t} \frac{\varepsilon_1}{\varepsilon_2} \rightarrow$ dependent on ε_2 also

i-e D_t undergoes some change across the interface. Hence D_t is discontinuous across the interface.

UNIT- III

MAXWELL'S EQUATIONS

UNIT – 3

References:

1. Balman & Jordan

consider fig:2 & eqn (2)

slide no: 3

$$dQ = \rho_s\, ds$$

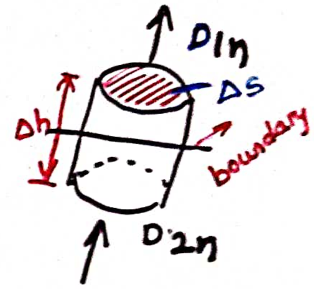
$$\Delta Q = \rho_s\, \Delta S.$$

$$\oint \vec{D}\cdot\vec{ds} = \oint_{top} \vec{D}\cdot\vec{ds} + \oint_{bottom} \vec{D}\cdot\vec{ds} + \oint_{Side} \vec{D}\cdot\vec{ds}$$

$$\oint_{Side} \vec{D}\cdot\vec{ds} = 0 \rightarrow \text{because } \Delta h \rightarrow 0$$

$$\oint \vec{D}\cdot\vec{ds} = \oint_{top} \vec{D}\cdot\vec{ds} + \oint_{bottom} \vec{D}\cdot\vec{ds} = Q_{enc}$$

Since the flux is Leaving the top and bottom Surfaces normally



$$D_{1n}\,\Delta S - D_{2n}\,\Delta S = Q$$

$$D_{1n}\,\Delta S - D_{2n}\,\Delta S = \rho_s\,\Delta S$$

$$\boxed{D_{1n} - D_{2n} = \rho_s}$$

If the surface charge is not present

$$\boxed{D_{1n} - D_{2n} = 0} \Rightarrow \boxed{D_{1n} = D_{2n}}$$

→ Normal component of $\vec{D}$ is continuous across the interface.

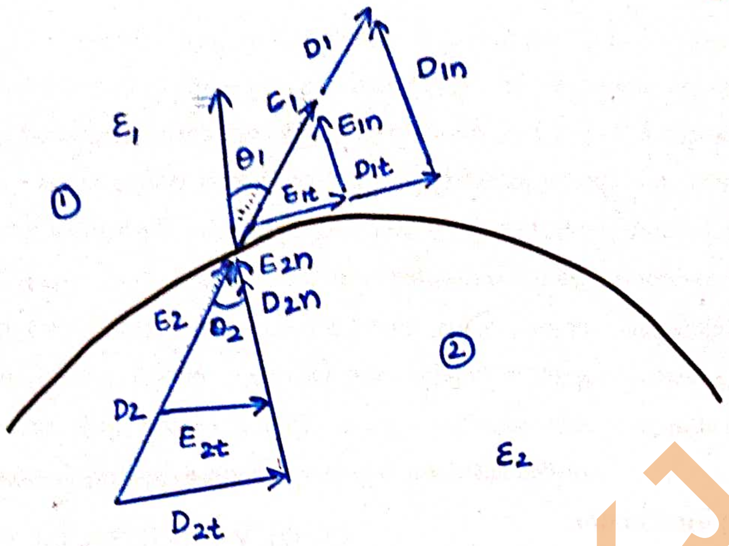
$$D = \varepsilon E$$

$$D_{1n} = \varepsilon_1 E_{1n} = \varepsilon_2 E_{2n} = D_{2n}$$

→ Normal component of $\vec{E}$ is discontinuous across the interface

Refraction of $\vec{D}$ (or) $\vec{E}$ at a dielectric - dielectric boundary.

slide no: 4



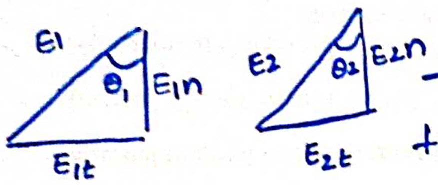


fig:1 fig:2

$\sin\theta_2 = \frac{E_{2t}}{E_2}$

$\rightarrow E_{2t} = E_2 \sin\theta_2$

$\sin\theta_1 = \frac{E_{1t}}{E_1} \Rightarrow E_{1t} = E_1 \sin\theta_1$

$D_{1n} = D_{2n}$ if $\rho_s = 0$

W.K.T $E_{t1} = E_{t2}$

$\boxed{E_1 \sin\theta_1 = E_2 \sin\theta_2}$ — ① $\boxed{\varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}}$

from fig 1: $\cos\theta_1 = \frac{E_{1n}}{E_1}$ $\cos\theta_2 = \frac{E_{2n}}{E_2}$

$E_{1n} = E_1 \cos\theta_1$ $E_{2n} = E_2 \cos\theta_2$

$\varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}$

$\boxed{\varepsilon_1 E_1 \cos\theta_1 = \varepsilon_2 E_2 \cos\theta_2} = D_{1n} = D_{2n}$ — ②

dividing ① by ② $\frac{\tan\theta_1}{\varepsilon_1} = \frac{\tan\theta_2}{\varepsilon_2}$

$$\boxed{\frac{\tan\theta_1}{\tan\theta_2} = \frac{\varepsilon_1}{\varepsilon_2} = \frac{\varepsilon_0 \varepsilon_{r1}}{\varepsilon_0 \varepsilon_{r2}} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}}$$

This is law of refraction of the $\vec{E}$ at the boundary

Conductor to dielectric Boundary conditions

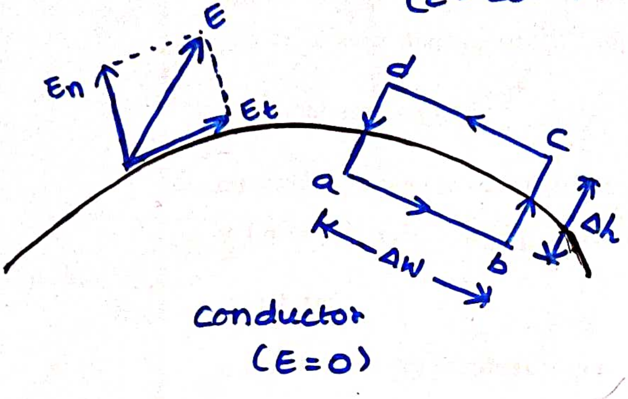
Slide no: 5

The conductor is assumed to be perfect

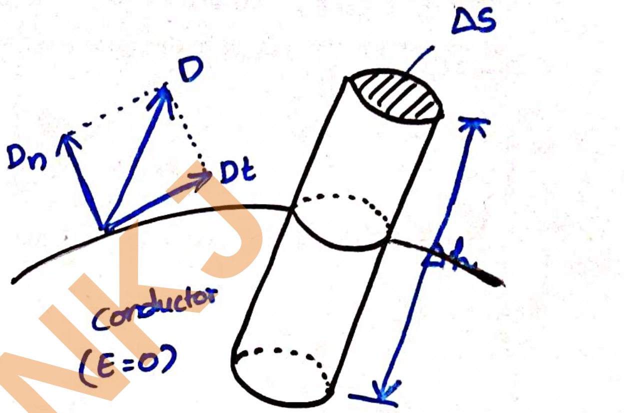
i.e $\sigma \to \infty$ (or) $\rho_c = 0$ $\quad J = \sigma E$

$E \propto \frac{1}{\sigma}$

$E = 0$ when $\sigma = \infty$



fig(a)



fig(b)

$$\oint \vec{E} \cdot \vec{dl} = 0$$

$$0 \cdot \Delta W + 0 \cdot \frac{\Delta h}{2} + E_n \frac{\Delta h}{2} - E_t \Delta W - E_n \frac{\Delta h}{2} - 0 \cdot \frac{\Delta h}{2}$$

As $\Delta h \to 0$

$\boxed{E_t = 0}$ tangential component of electric field is zero.

from fig (b)

$$\Delta Q = D_n \Delta S - 0 \cdot \Delta S = \rho_S \Delta S$$

$$\boxed{D_n = \rho_S}$$

$\boxed{D_t = \varepsilon_0 \varepsilon_r E_t = 0}$, $\quad \boxed{D_n = \varepsilon_0 \varepsilon_r E_n = \rho_S}$

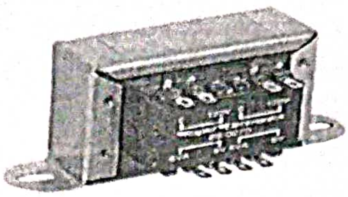


Fig no:3.1.1 An Electrical Transformer

Turns ratio = $V_p/V_S = N_p/N_S$

Power Out= Power In

$V_S \times I_S = V_P \times I_P$

Vp=primary(input)voltage Np=number of turns on primarycoil

Ip = primary (input) current

3.2 RECTIFIER:

A circuit, which is used to convert a.c to dc, is known as RECTIFIER. The process of conversion a.c to d.c is called "rectification"

3.2.1 TYPES OF RECTIFIERS:

- Half wave Rectifier
- Full wave rectifier

1. Center tap full wave rectifier.
2. Bridge type full bridge rectifier.

3.3 Filter:

A Filter is a device, which removes the a.c component of rectifier output but allows the d.c component to reach the load

3.3.1 Capacitor Filter:

We have seen that the ripple content in the rectified output of half wave rectifier is **121%** or that of full-wave or bridge rectifier or bridge rectifier is **48%** such high percentages of ripples is not acceptable for most of the applications. Ripples can be removed by one of the following methods of filtering:

(a) A capacitor, in parallel to the load, provides an easier by –pass for the ripples voltage though it due to low impedance. At ripple frequency and leave the d.c.to appears the load.

Boundary conditions in Magnetic fields

Maxwell's equations $\rightarrow$ $\oint \vec{H}\cdot\vec{dl} = I_{enc}$ & $\oint_S \vec{B}\cdot\vec{ds} = 0$

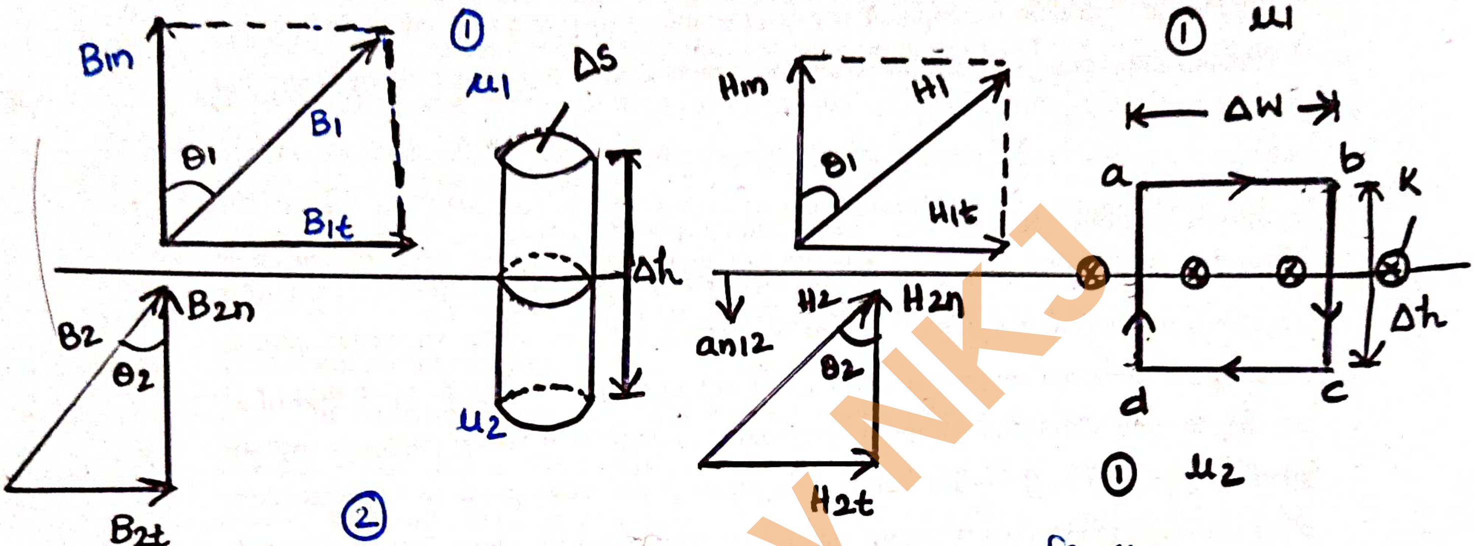


fig (a)   fig (b)

Fig: Boundary conditions between two magnetic media (a) for $\vec{B}$, (b) for $\vec{H}$.

Consider fig (1) and eqn $\oint_S \vec{B}\cdot\vec{ds} = 0$

$$= \oint_{S\,top} \vec{B}\cdot\vec{ds} + \oint_{side} \vec{B}\cdot\vec{ds} + \oint_{bottom} \vec{B}\cdot\vec{ds}$$

As $\Delta h \rightarrow 0$
$\oint_{side} \vec{B}\cdot\vec{ds} = 0$

$$\Rightarrow \quad B_{1n}\,\Delta S - B_{2n}\,\Delta S = 0$$

$$\boxed{B_{1n} = B_{2n}}$$ $\rightarrow$ Normal component of $\vec{B}$ is continuous at the boundary

$\Rightarrow \mu_1 H_{1n} = \mu_2 H_{2n}$

(or) $H_{1n} = \frac{\mu_2}{\mu_1} H_{2n}$ $\rightarrow$ This shows that Normal component of $\vec{H}$ is discontinuous at the boundary.

from fig (2) and eqn $\oint \vec{H} \cdot \vec{dl} = I$ is applied to the closed path abcda of fig (2) where surface current 'K' on the boundary is assumed normal to the path.

Surface current density = current / unit width

$$K = I/\Delta w \Rightarrow I = K\Delta w$$

$$K\Delta w = H_{1t}\Delta w - H_{1n}\frac{\Delta h}{2} - H_{2n}\frac{\Delta h}{2} - H_{2t}\Delta w + H_{2n}\frac{\Delta h}{2} + H_{1n}\frac{\Delta h}{2}$$

As $\Delta h \to 0$ $\boxed{K = H_{1t} - H_{2t}}$ depends on Surface Current.

This shows that the tangential component of $\vec{H}$ is discontinuous

$$\boxed{K = \frac{B_{1t}}{\mu_1} - \frac{B_{2t}}{\mu_2}}$$

If the boundary is free of current (or) media are not conductors

$K = 0 \Rightarrow \boxed{H_{1t} = H_{2t}} \rightarrow$ Continuous

$\vec{B}$ is discontinuous

from fig (1)

$$\cos\theta_1 = \frac{B_{1n}}{B_1} \Rightarrow B_{1n} = B_1\cos\theta_1$$

$$\cos\theta_2 = \frac{B_{2n}}{B_2} \Rightarrow B_{2n} = B_2\cos\theta_2$$

$\boxed{B_{1n} = B_{2n}} \Rightarrow B_1\cos\theta_1 = B_2\cos\theta_2$ — (1)

$$\sin\theta_1 = \frac{B_{1t}}{B_1} \Rightarrow B_{1t} = B_1\sin\theta_1$$

$\frac{B_{1t}}{\mu_1} = \frac{B_{2t}}{\mu_2}$ when $K = 0$

$$\sin\theta_2 = \frac{B_{2t}}{B_2} \Rightarrow B_{2t} = B_2\sin\theta_2$$ — (2)

divide (2) by (1) we get $\frac{B_1\sin\theta_1}{\mu_1} = \frac{B_2\sin\theta_2}{\mu_2}$

$$\boxed{\frac{\tan\theta_1}{\tan\theta_2} = \frac{\mu_1}{\mu_2}}$$

Energy stored in the Magnetic field:

$$V = L\frac{dI}{dt} \quad , \quad P = VI = LI\frac{dI}{dt} \quad \text{--- (1)}$$

(or) $P\,dt = LI\,dI$

The time integral of power is Work (or Energy)

$$W_m = \int_0^t P\,dt = \int_0^I LI\,dI = \frac{1}{2}LI^2$$

Power = Energy per unit time

$P = \frac{E}{t} = \frac{dE}{dt}$.

Substitute (1) i.e P $\int LI\frac{dI}{dt} \times dt$

$= \int LI\,dI = \frac{1}{2}LI^2$ ✓

$$W_m = \frac{1}{2}LI^2 \text{ Joules}$$

Note : The inductance of an inductor can also be defined as $L = \frac{2W_m}{I^2}$ Henry's

Mutual inductance:

When two coils are coupled together i.e they are kept in a common magnetic field as shown in the fig:

The flux produced by one coil Links with that of the other and as a result, Voltage is generated in the second coil.

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The Mutual inductance (M) between two coils is defined as the ratio of total flux Linkage in one Coil to the current in the other coil.

Energy density in Electrostatic fields :-

The energy (or) workdone in an electric field due to 'N' point charges is expressed as

$$W_E = \frac{1}{2}\sum_{k=1}^{n} Q_k V_k \text{ in Joules}$$

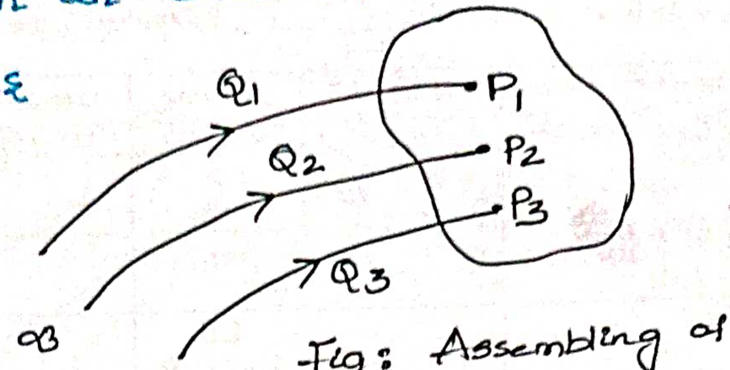

Fig: Assembling of charges

Consider a freespace where there is no electric field. Let a point charge 'Q_1' be moved from infinity to a point P_1 in the freespace as shown in fig:

Since there is no electric field (initially) the workdone is zero i.e $\boxed{W_1 = 0}$

$$E = 0$$

$$V = \frac{W}{Q} \qquad V = -\int \vec{E}\cdot\vec{dl} = 0$$

$$W = VQ = 0$$

Now due to 'Q_1' in the freespace an electric field is established.

If a point charge 'Q_2' is moved from '∞' to a point 'P_2' in this field, then the workdone in keeping 'Q_2' is

$$\boxed{W_2 = Q_2 V_{21}}$$

where V_{21} is the potential at point '2' due to 'Q_1'

$|||^{ly}$ If a point charge 'Q_3' is moved from ∞ to a point P_3 then the workdone in keeping Q_3 is

$$\boxed{W_3 = Q_3 V_{31} + Q_3 V_{32}}$$

where V_{31} is the potential at point '3' due to 'Q_1' &
V_{32} " " " " " " Q_2.

The total workdone in positioning (keeping) the three point charges is $W_E = W_1 + W_2 + W_3$

$$= 0 + Q_2 V_{21} + Q_3 (V_{31} + V_{32}) \quad \text{--- (1)}$$

If the charges were positioned in reverse order

$$W_E = W_3 + W_2 + W_1$$
$$= 0 + Q_2 V_{23} + Q_1 (V_{12} + V_{13}) \quad \text{--- (2)}$$

Where V_{23} is the potential at point 2 due to Q_3 .....

Add (1) & (2)

$$2W_E = Q_2 (V_{21} + V_{23}) + Q_1 (V_{12} + V_{13}) + Q_3 (V_{31} + V_{32})$$

$$2W_E = Q_2 V_2 + Q_1 V_1 + Q_3 V_3$$

$$W_E = \frac{1}{2} [Q_2 V_2 + Q_1 V_1 + Q_3 V_3]$$

Where V_1, V_2 & V_3 are total potentials at P_1, P_2 & P_3 respectively. Ingeneral if there are 'n' point charges

$$\boxed{W_E = \frac{1}{2} \sum_{k=1}^{n} Q_k V_k \quad \text{Joules}}$$

Energy stored in the capacitor is $W_E = \frac{1}{2} C V^2$

Amount of energy stored in a capacitor can be found by calculating the workdone in charging a capacitor.

$$V = \frac{W}{Q} \Rightarrow W = VQ$$

Workdone in moving a small charge dQ against a potential difference 'V' is

$$Q = CV$$

$$dw = V dQ$$
$$dw = \frac{Q}{C} dQ$$

Total workdone in charging a capacitor to 'Q' coulomb's is

$$W = \int_0^Q \frac{Q}{C} dQ = \frac{Q^2}{2C} = \frac{C^2 V^2}{2C} = \frac{1}{2} C V^2$$

$$W = \frac{1}{2} C V^2 \qquad C = \frac{\varepsilon S}{d}$$

$$\Delta W = \frac{1}{2} \Delta C (\Delta V)^2 \qquad \Delta C = \frac{\varepsilon \Delta S}{\Delta d}$$

$$= \frac{1}{2} \varepsilon \frac{\Delta S}{\Delta d} E^2 \times (\Delta d)^2$$

$$= \frac{1}{2} \varepsilon E^2 \Delta S \times \Delta d \qquad E = \frac{V}{d}$$

$$= \frac{1}{2} \varepsilon E^2 \Delta V \qquad E = \frac{\Delta V}{\Delta d}$$

$E \downarrow$ static Electric field

$$W = \int \frac{1}{2} \varepsilon E^2 dv = \frac{1}{2} \int \vec{D} \cdot \vec{E} \, dv$$

Electrostatic Energy density ω_E (in J/m^3) as

workdone per unit volume

$$\omega_E = \frac{d(W)}{dv}$$

$$= \frac{d}{dv} \int \frac{1}{2} \varepsilon E^2 dv = \frac{1}{2} \varepsilon E^2$$

(or)

$$W_E = \int \omega_E \, dv \qquad = \frac{1}{2} \vec{D} \cdot \vec{E}$$

RC time constant is equal to the relaxation time:

$$V = IR \qquad C = Q/V$$

$$R = \frac{V}{I} \implies RC = \frac{\cancel{V}}{I} \times \frac{Q}{\cancel{V}} = \frac{Q}{I}$$

$$Q = \int_v \rho_v dv, \quad I = \int_S \vec{J} \cdot \vec{ds}$$

$$RC = \frac{\int_v \rho_v dv}{\int_S \vec{J} \cdot \vec{ds}} = \frac{\int_v (\nabla \cdot D) dv}{\int_S \vec{J} \cdot \vec{ds}} = \frac{\int_S \vec{D} \cdot \vec{ds}}{\int_S \vec{J} \cdot \vec{ds}} = \frac{\vec{D}}{\vec{J}}$$

$$RC = \frac{\varepsilon \vec{E}}{\sigma \vec{E}} = \frac{\varepsilon}{\sigma} = T_r$$

Unit - 3

EM Wave characteristics - I

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EM WAVE CHARECTERISTICS:-1

→ The four Maxwell's equations for time varying fields are

$$\nabla\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t} \Rightarrow \vec{J} + \frac{d\vec{D}}{dt} = J + \dot{D} \quad —(1)$$

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = -\dot{B} \quad —(2)$$

$$\nabla\cdot\vec{D} = \rho_v \quad —(3)$$

$$\nabla\cdot\vec{B} = 0 \quad —(4)$$

waves are means of transporting Energy (or) information

in which the dot superscript indicates partial differentiation with respect to time.

A wave is a function of both space & time.

and the relations are $\vec{D} = \varepsilon\vec{E}$ (or) $\dot{D} = \varepsilon\dot{E}$

$\vec{B} = \mu\vec{H}$ (or) $\dot{B} = \mu\dot{H}$

$\vec{J} = \sigma\vec{E}$ (or) $\dot{J} = \sigma\dot{E}$

Wave equations :-

(i) For Free space :

* $\nabla\times H = J + \dot{D}$

In freespace (or) more generally in a perfect dielectric containing no charges and no conduction currents. i.e $\boxed{J=0}$

$\nabla\cdot\vec{J} = -\frac{d\rho_v}{dt} = 0$

$\int_s \vec{J}\cdot d\vec{s} = 0$

$\vec{J} = 0$

$\nabla\times H = \dot{D}$

$\nabla\times H = \varepsilon\dot{E} \quad (\because \dot{D} = \varepsilon\dot{E})$

$\nabla\times E = -\dot{B}$

$\nabla\times\dot{E} = -\ddot{B} = \frac{\partial^2\vec{B}}{\partial t^2}$

$\nabla\times(\nabla\times H) = \nabla\times\varepsilon\dot{E} = \varepsilon(\nabla\times\dot{E})$

$= \varepsilon(-\ddot{B})$

$= \varepsilon(-\mu\ddot{H})$

The Vector identity

W.K.T

$\nabla\cdot B = 0$

$\nabla\cdot\mu H = 0$

$\nabla\cdot H = 0$

$\nabla\times(\nabla\times H) = \nabla(\nabla\cdot H) - \nabla^2 H = -\mu\varepsilon\ddot{H}$

$\Rightarrow -\nabla^2 H = -\mu\varepsilon\ddot{H}$

$$\boxed{\nabla^2 H = \mu\varepsilon\ddot{H}}$$

*** This is free space wave equation for magnetic fields.

* $\nabla\times E = -\dot{B}$

$= -\mu\dot{H}$

In free space

$\nabla\times H = \dot{D}$

$\nabla\times\dot{H} = \ddot{D}$

$\nabla\times(\nabla\times E) = -\mu(\nabla\times\dot{H})$

$= -\mu(\ddot{D})$

$$\nabla \times (\nabla \times E) = -\mu(\ddot{D})$$

$$\nabla \times (\nabla \times E) = -\mu(\varepsilon \ddot{E}) \quad (\therefore D = \varepsilon E)$$

$$\nabla(\nabla \cdot E) - \nabla^2 E = -\mu\varepsilon\ddot{E}$$

[$\because \nabla \cdot D = \rho_v$ space current = 0 for free charge = 0]

$$\nabla\left(\frac{\nabla \cdot D}{\varepsilon}\right) - \nabla^2 E = -\mu\varepsilon\ddot{E}$$

$\nabla \cdot D = 0$

$$\boxed{\nabla^2 E = \mu\varepsilon\ddot{E}}$$

This is the free space wave equation for electric fields.

ii) For conducting medium

* For regions in which the conductivity is not zero ($\neq 0$) and conduction current exists, a more general solution must be obtained

W.K.T $\quad \nabla \times H = J + \dot{D} \qquad J = \sigma E$ & $D = \varepsilon E$

$$\nabla \times H = \sigma E + \varepsilon \dot{E}$$

W.K.T $\quad \nabla \times E = -\dot{B} = -\mu\dot{H}$

$$\nabla \times (\nabla \times E) = -\mu(\nabla \times \dot{H})$$

$$= -\mu(\sigma\dot{E} + \varepsilon\ddot{E})$$

$$= -\mu\varepsilon\ddot{E} - \mu\sigma\dot{E}$$

$$\nabla \times (\nabla \times E) = \nabla(\nabla \cdot E) - \nabla^2 E = -\mu\varepsilon\ddot{E} - \mu\sigma\dot{E}$$

$$\Rightarrow \nabla\left(\frac{\nabla \cdot D}{\varepsilon}\right) - \nabla^2 E = -\mu\varepsilon\ddot{E} - \mu\sigma\dot{E}$$

$$\nabla^2 E = \mu\varepsilon\ddot{E} + \mu\sigma\dot{E} \quad \text{(or)}$$

$$\nabla^2 E - (\mu\varepsilon\ddot{E} + \mu\sigma\dot{E}) = 0 \quad \text{(or)}$$

$$\boxed{\nabla^2 E - \mu\varepsilon\ddot{E} - \mu\sigma\dot{E} = 0}$$

$\nabla\left(\frac{\nabla \cdot D}{\varepsilon}\right)$

$\frac{1}{\varepsilon}\nabla(\nabla \cdot D)$

$\frac{1}{\varepsilon}\nabla(\rho_v)$

$\frac{1}{\varepsilon}\left[\frac{\partial}{\partial x}(\rho_v) + \frac{\partial}{\partial y}(\rho_v) + \frac{\partial}{\partial z}(\rho_v)\right]$

$= 0$

This is the wave equation for electric field (E) in conducting medium

* $\nabla \times H = J + \dot{D}$

$$\nabla \times (\nabla \times H) = \nabla \times (J + \dot{D}) = \nabla \times (\sigma E + \varepsilon \dot{E})$$

$$= \sigma(\nabla \times E) + \varepsilon(\nabla \times \dot{E})$$

$$\nabla\times(\nabla\times H) = \sigma(\nabla\times E) + \varepsilon(\nabla\times \dot{E})$$

from eqn ②

$\nabla\times E = -\dot{B}$

$= -\mu\dot{H}$

$$\nabla(\nabla\cdot H) - \nabla^2 H = \sigma(-\mu\dot{H}) + \varepsilon(-\mu\ddot{H})$$

$$\nabla\left(\frac{\nabla\cdot B}{\mu}\right) - \nabla^2 H = -\mu\sigma\dot{H} - \mu\varepsilon\ddot{H}$$

$B = \mu H$

$H = \frac{1}{\mu}B$

$$\boxed{\nabla^2 H - \mu\sigma\dot{H} - \mu\varepsilon\ddot{H} = 0}$$

$\nabla\cdot B = 0$

This is the wave equation for magnetic field (H)

Uniform plane waves :-

See - Backside

The wave equation reduces to a very simple form in the special case where E and H are considered to be independent of two dimensions, say y and z. Then the plane-wave equation is

$$\nabla^2 E = \mu\varepsilon\ddot{E} \Rightarrow \frac{\partial^2 E}{\partial x^2} = \mu\varepsilon\frac{\partial^2 E}{\partial t^2}$$

This is the free space wave eqn for electric fields.

may be written in terms of the components of $\vec{E}$ as

$$\frac{\partial^2 E_x}{\partial x^2} = \mu\varepsilon\frac{\partial^2 E_x}{\partial t^2}$$

$$\frac{\partial^2 E_y}{\partial y^2} = \mu\varepsilon\frac{\partial^2 E_y}{\partial t^2}$$

$$\frac{\partial^2 E_z}{\partial z^2} = \mu\varepsilon\frac{\partial^2 E_z}{\partial t^2}$$

*** Where the electric field is independent of y and z and is a function of x and t only. Such a wave is called a uniform plane wave.

In a region in which there is no charge density i.e. $\rho_v = 0$

$$\nabla\cdot E = \frac{1}{\varepsilon}\nabla\cdot D = \rho_v = 0$$

In a homogeneous medium ε is constant.

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0$$

For a uniform plane wave in which $\vec{E}$ is independent of y and z the last two terms of this relation are equal to zero. So that it reduces to

$$\boxed{\frac{\partial E_x}{\partial x} = 0}$$

Therefore there is no variation of E_x in the x-direction.

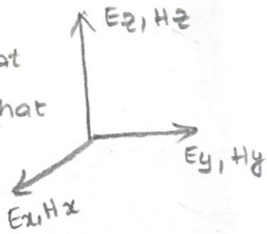
& from eqn $\frac{\partial E_x}{\partial x} = 0$, it is seen that the second derivative w.r.to time of E_x must then be zero. This requires that E_x be either zero, constant in time, (or) increasing uniformly with time.

Therefore a uniform plane wave progressing (propagating (or) travelling) in the x-direction has no 'x' component of $\vec{E}$.

Similarly $\nabla \cdot \vec{B} = 0 = \nabla \cdot \mu \vec{H} = 0 \Rightarrow \nabla \cdot \vec{H} = 0$ we can proove that $\boxed{\frac{\partial H_x}{\partial x} = 0}$ shows that there is no variation of H_x in the x-direction.

It follows, therefore, that uniform plane electromagnetic waves are tranverse and have components of $\vec{E}$ and $\vec{H}$ only in directions perpendicular to the direction of propagation.

Assuming that $\vec{E}$ is independent of y and z direction that any $\vec{E}$ and $\vec{H}$ having (travelling) only x direction & also $E_x = H_x = 0$ indicates that $\vec{E}$ and $\vec{H}$ having components only in the direction of y & z. and these directions are ⊥ar to the direction of propagation.



Relation between E and H :-

$$\nabla \times E = \begin{vmatrix} \vec{x} & \vec{y} & \vec{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

For a uniform plane wave propagating along x-direction $E_x = 0$, $\frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0$

$\frac{\partial}{\partial x}$ = change of wave w.r.to 'x'

$$\nabla \times E = \begin{vmatrix} \vec{x} & \vec{y} & \vec{z} \\ \frac{\partial}{\partial x} & 0 & 0 \\ 0 & E_y & E_z \end{vmatrix} = \vec{z}\left(\frac{\partial}{\partial x} E_y\right) \ominus \vec{y}\left(\frac{\partial}{\partial x} E_z\right)$$

$$= -\frac{\partial E_z}{\partial x}\vec{y} + \frac{\partial E_y}{\partial x}\vec{z}$$

$\frac{\partial}{\partial y} = \frac{\partial}{\partial z}$ = nochange $= 0$

III^{ly}

$$\nabla \times H = -\vec{y}\frac{\partial H_z}{\partial x} + \frac{\partial H_y}{\partial x}\vec{z}$$

$$\nabla \times E = -\frac{\partial E_z}{\partial x}\vec{y} + \frac{\partial E_y}{\partial x}\vec{z} \quad — (1)$$

$$\nabla \times H = -\frac{\partial H_z}{\partial x}\vec{y} + \frac{\partial H_y}{\partial x}\vec{z} \quad — (2)$$

But $\nabla \times H = \dot{D}$ (for free space)

$$= \varepsilon \dot{E} \quad (\because D = \varepsilon E)$$

$$= \varepsilon \frac{\partial E}{\partial t}$$

flux ψ in terms of Magnetic vector Potential

$$\vec{E} = E_x \vec{a_x} + E_y \vec{a_y} + E_z \vec{a_z}$$

Now $\nabla \times H = \varepsilon \frac{\partial}{\partial t}\left[E_x \vec{a_x} + E_y \vec{a_y} + E_z \vec{a_z}\right]$

For uniform plane wave $E_x = 0$ The above eqn is reduced to

$$\nabla \times H = \varepsilon\left[\frac{\partial E_y}{\partial t}\vec{y} + \frac{\partial E_z}{\partial t}\vec{z}\right] \quad — (3)$$

comparing eqns (2) & (3) we get

$$-\frac{\partial H_z}{\partial x}\vec{y} + \frac{\partial H_y}{\partial x}\vec{z} = \varepsilon \frac{\partial E_y}{\partial t}\vec{y} + \frac{\partial E_z}{\partial t}\vec{z}$$

Equating the $\vec{y}$ terms and $\vec{z}$ terms

$$-\frac{\partial H_z}{\partial x} = \varepsilon \frac{\partial E_y}{\partial t} \quad — (4)$$

$$\frac{\partial H_y}{\partial x} = \varepsilon \frac{\partial E_z}{\partial t} \quad — (5)$$

Similarly $\nabla \times E = -\dot{B} = -\mu \dot{H} = -\mu \frac{\partial H}{\partial t} = -\mu\left[\frac{\partial H_y}{\partial t}\vec{y} + \frac{\partial H_z}{\partial t}\vec{z}\right] \quad (\because H_x = 0)$

eqn (1)

$$-\frac{\partial E_z}{\partial x}\vec{y} + \frac{\partial E_y}{\partial x}\vec{z} = -\mu\left[\frac{\partial H_y}{\partial t}\vec{y} + \frac{\partial H_z}{\partial t}\vec{z}\right]$$

$$+\frac{\partial E_z}{\partial x} = +\mu \frac{\partial H_y}{\partial t} \quad — (6)$$

$$\frac{\partial E_y}{\partial x} = -\mu \frac{\partial H_z}{\partial t} \quad — (7)$$

$f_1(x - v_0 t)$

a function f_1 of the variable $(x - v_0 t)$

If $E_y = f_1(x - v_0 t)$ where $v_0 = \frac{1}{\sqrt{\mu \varepsilon}}$ then

$f_1(x - v_0 t) = F$

$\frac{\partial E_y}{\partial t} = f'(x - v_0 t) \cdot \frac{d}{dt}(x - v_0 t)$

$$\frac{\partial E_y}{\partial t} = \frac{\partial f_1(x - v_0 t)}{\partial (x - v_0 t)} \times \frac{d(x - v_0 t)}{dt}$$

$= -v_0 f_1'(x - v_0 t)$

$= -v_0 F'$

$$= -v_0 \cdot \frac{\partial f_1(x - v_0 t)}{\partial (x - v_0 t)} = -v_0 f_1' \quad — (8)$$

Now Substituting $\frac{\partial E_y}{\partial t} = -v_0 f_1'$ in eqn (4)

$$-\frac{\partial H_z}{\partial x} = \varepsilon \frac{\partial E_y}{\partial t} = -v_0 \varepsilon f_1' \quad \text{(or)} \quad \frac{\partial H_z}{\partial x} = v_0 \varepsilon f_1'$$

Then

$$\frac{\partial H_z}{\partial x} = \frac{1}{\sqrt{\mu\varepsilon}} \times \varepsilon f_1' = \sqrt{\frac{\varepsilon}{\mu}} f_1'$$

$$f_1' = \frac{\partial f_1(x - v_0 t)}{\partial (x - v_0 t)}$$

$$H_z = \sqrt{\frac{\varepsilon}{\mu}} \int f_1' \, dx + C$$

Now

$$\frac{\partial (f_1)}{\partial x} = \frac{\partial f_1(x - v_0 t)}{\partial (x - v_0 t)} \times \frac{\partial (x - v_0 t)}{\partial x} = f_1'$$

$$= \sqrt{\frac{\varepsilon}{\mu}} f_1 + C = \sqrt{\frac{\varepsilon}{\mu}} E_y + C$$

C indicates that the field independent of x would be present. Therefore 'C' can be neglected.

$$H_z = \sqrt{\frac{\varepsilon}{\mu}} E_y \quad \text{(or)} \quad \boxed{\frac{E_y}{H_z} = \sqrt{\frac{\mu}{\varepsilon}}} \quad — (9)$$

Similarly from eqn (4) & (5)

$$\boxed{\frac{E_z}{H_y} = -\sqrt{\frac{\mu}{\varepsilon}}} \quad — (10)$$

$$E_z = f_1(x - v_0 t)$$

$$\frac{\partial E_z}{\partial t} = -v_0 F'$$

$$\varepsilon \frac{\partial E_z}{\partial z} = -v_0 F' \varepsilon$$

$$H_y = \int \varepsilon (-v_0 F')$$

$$= -\varepsilon v_0 F$$

$$= -\varepsilon v_0 E_z$$

$$\frac{E_z}{H_y} = \frac{-1}{\varepsilon v_0} = -\sqrt{\frac{\mu}{\varepsilon}}$$

Since $E = \sqrt{E_z^2 + E_y^2}$ and $H = \sqrt{H_y^2 + H_z^2}$

Squaring both sides of Eqn (9) & (10)

$$E_y^2 = H_z^2 \sqrt{\frac{\mu}{\varepsilon}} \qquad E_z^2 = H_y^2 \sqrt{\frac{\mu}{\varepsilon}}$$

$$E^2 = E_y^2 + E_z^2 = H_z^2 \sqrt{\frac{\mu}{\varepsilon}} + H_y^2 \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}} \left[H_y^2 + H_z^2 \right]$$

$$E^2 = \sqrt{\frac{\mu}{\varepsilon}} H^2 \Rightarrow \boxed{\frac{E}{H} = \sqrt{\frac{\mu}{\varepsilon}}}$$

$$\frac{\text{Volt/amp}}{\text{current/amp}} = \text{Impedance}$$

$\frac{E}{H}$ has the units of impedance (or) ohms. For this reason the ratio $\sqrt{\frac{\mu}{\varepsilon}}$ is referred to as the characteristic impedance (or) intrinsic impedance of the non-conducting medium. For free space

$$\mu_r = \varepsilon_r = 1 \qquad \mu_0 = 4\pi \times 10^{-7} \; ; \; \varepsilon_0 = \frac{1}{36\pi \times 10^9}$$

intrinsic impedance is designated by the symbol 'η'.

The square root of the ratio of Permeability to the permittivity is called the intrinsic impedance η (eta)

$$\eta = \sqrt{\frac{\mu_0\mu_r}{\varepsilon_0\varepsilon_r}} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \sqrt{\frac{4\pi\times10^{-7}}{\frac{1}{36\pi\times10^{9}}}} = \boxed{377\,\Omega = 120\pi}$$

dot product of $\vec{E}$ & $\vec{H}$

$\vec{E} = \vec{x}E_x + \vec{y}E_y + \vec{z}E_z$; $\vec{H} = \vec{x}H_x + \vec{y}H_y + \vec{z}H_z$

$\vec{E}\cdot\vec{H} = E_xH_x + E_yH_y + E_zH_z$ but for uniform plane wave $E_x = H_x = 0$

$\vec{E}\cdot\vec{H} = E_yH_y + E_zH_z$ from eqns (9) & (10)

$E_y = \eta H_z$; $E_z = -\eta H_y$ substitute in the above eqn we get

$\vec{E}\cdot\vec{H} = \eta H_zH_y - \eta H_zH_y = 0 \rightarrow$ the angle between E & H is $90°$

$|E||H|\cos\theta = 0$ Thus in a uniform plane wave, E and H are at right angles to each other (or) perpendicular to each other.

Sinusoidal time variations :

→ Most generators produce voltages and currents which are sinusoidal. therefore E and H fields are also sinusoidally time invariant. This can be expressed as

$$E = E_0\cos\omega t$$
$$E = E_0\sin\omega t$$

where $f = \frac{\omega}{2\pi}$ is the frequency of the variation.

→ The time varying field $\tilde{E}(r,t)$ may be expressed in terms of the corresponding phasor quantity $E(r)$ as

$$\tilde{E}(r,t) = Re\{E(r)\,e^{j\omega t}\}$$

$\sim \rightarrow$ time varying field.

For ex :

→ The phasor E_x is defined by the relation

$$\tilde{E}_x(r,t) = Re\{E_x(r)\,e^{j\omega t}\}$$

Where $e^{j\omega t} = \cos\omega t + j\sin\omega t$.

* In the phasor domain, time derivatives become multiplication by $j\omega$. i.e

$$\frac{d}{dt}\tilde{E}(r,t) = Re\{E(r)\,j\omega\,e^{j\omega t}\}$$

Maxwell's Equations Using Phasor Notation:

consider 1^{st} Maxwell equation in time varying fields

$$\nabla \times \tilde{H} = \frac{d\tilde{D}}{dt} + \tilde{J}$$

In Phasor form

$$\nabla \times \mathrm{Re}\{H e^{j\omega t}\} = \frac{d}{dt}\mathrm{Re}\{D e^{j\omega t}\} + \mathrm{Re}\{J e^{j\omega t}\}$$

$$\mathrm{Re}\left\{\{\nabla \times H - j\omega D - J\} e^{j\omega t}\right\} = 0$$

This eqn is valid for all values of 't'

then $\boxed{\nabla \times H = j\omega D + J}$

The phasor equation may be derived from the time-varying equation by replacing each time-varying quantity with a phasor quantity and each time derivative with a $j\omega$ factor.

$$\nabla \times H = J + \frac{dD}{dt} \qquad \nabla \times H = J + j\omega D \qquad \oint H\cdot ds = \int (j\omega D + J)\cdot ds$$

$$\nabla \times E = -\frac{dB}{dt} \qquad \nabla \times E = -j\omega B \qquad \oint_s E\cdot ds = -\int j\omega B\cdot d\vec{a}$$

$$\nabla \cdot D = \rho_v \qquad \oint D\cdot ds = \int \rho\, dv$$

$$\nabla \cdot B = 0 \qquad \oint B\cdot ds = 0$$

$$\nabla \cdot J = -\frac{d\rho_v}{dt} \qquad \nabla \cdot J = -j\omega\rho_v \qquad \oint J\cdot ds = -\int j\omega\rho\, dv.$$

Wave equations in phasor Notation:-

For freespace

$$\nabla^2 E = \mu\varepsilon \ddot{E}$$
$$= \mu\varepsilon (j\omega)(j\omega) E$$
$$= -\omega^2 \mu\varepsilon E \rightarrow$$ *** called as Vector Helmholtz eqn

$$\nabla^2 H = \mu\varepsilon \ddot{H}$$
$$= -\omega^2\mu\varepsilon H$$

In conducting medium

$$\nabla^2 E - \mu\varepsilon \ddot{E} - \mu\sigma \dot{E} = 0$$
$$\nabla^2 E - \mu\varepsilon (j\omega)(j\omega) E - \mu\sigma j\omega E = 0$$
$$\nabla^2 E + \mu\varepsilon E\omega^2 - j\omega\mu\sigma E = 0$$
$$\nabla^2 E + (\omega^2\mu\varepsilon - j\omega\mu\sigma) E = 0$$

III'ly

$$\nabla^2 H + (\omega^2\mu\varepsilon - j\omega\mu\sigma) H = 0$$

Wave Propagating in a Lossless Medium

Consider the wave eqn in phasor form

consider uniform plane wave along x-direction

$$\nabla^2 E = -\omega^2 \mu \varepsilon E$$

$$\frac{d^2 E}{dx^2} = -\beta^2 E$$

where $\beta = \omega\sqrt{\mu\varepsilon}$ rad/m is called the phase shift constant.

Now taking the y-component of $\vec{E}$ only $\frac{d^2 E_y}{dx^2} = -\beta^2 E_y$ The solution of the wave equation is

$$E_y = A_1 e^{-j\beta x} + A_2 e^{j\beta x}$$

$D^2 + \beta^2 = 0$

$D^2 = -\beta^2$

$D^2 = (j\beta)^2$

$D = \pm j\beta$

Where A_1 and A_2 are arbitrary complex constants.

The corresponding time-varying field is

$$\tilde{E}_y(x,t) = Re\{E_y(x)\, e^{j\omega t}\}$$

$e^{j\omega t} = \cos\omega t + j\sin\omega t$

$$= Re\{\{A_1 e^{-j\beta x} + A_2 e^{j\beta x}\} e^{j\omega t}\}$$

$$= Re\{A_1 e^{j(\omega t-\beta x)} + A_2 e^{j(\omega t+\beta x)}\}$$

$$= Re\{A_1\{\cos(\omega t-\beta x) + j\sin(\omega t-\beta x)\} + A_2\{\cos(\omega t+\beta x) + j\sin(\omega t+\beta x)\}\}$$

By taking the real part of the above eqn

$$\tilde{E}_y(x,t) = A_1\cos(\omega t-\beta x) + A_2\cos(\omega t+\beta x)$$

By taking the imaginary part of the above eqn

$$\tilde{E}_y(x,t) = A_1\sin(\omega t-\beta x) + A_2\sin(\omega t+\beta x)$$

E^+ E^-

$E^- \rightarrow$ wave travelling in the $-x$ direction

Let $E = A\sin(\omega t - \beta x)$ is travelling with a velocity 'v' in the x-direction. Now consider a fixed point 'P' on the wave as shown in fig:

→ A is called the amplitude of the wave and has the same units as E

→ $(\omega t - \beta x)$ is the phase (in radians) of the wave; it depends on time 't' and variable 'x'.

Point 'P' is a Point of constant phase therefore

$$\omega t - \beta x = \text{a constant}$$

For a wave travelling in the +ve x-direction the point 'P' is given by $\omega t - \beta x$ = constant

$$(\text{or}) \quad \boxed{\frac{dx}{dt} = \frac{\omega}{\beta} = v}$$

Backside.

This Velocity of some point in the sinusoidal waveform is called the Phase velocity and β is called the phase shift constant and is a measure of the phase shift in radians per unit length (rad/m)

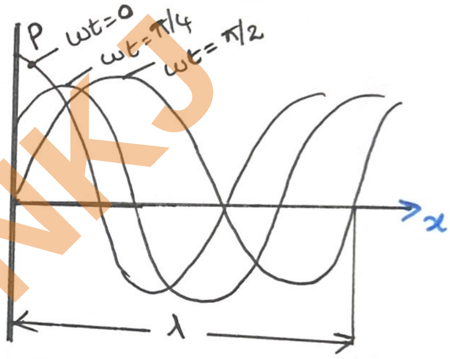


fig: Sinusoidal travelling wave

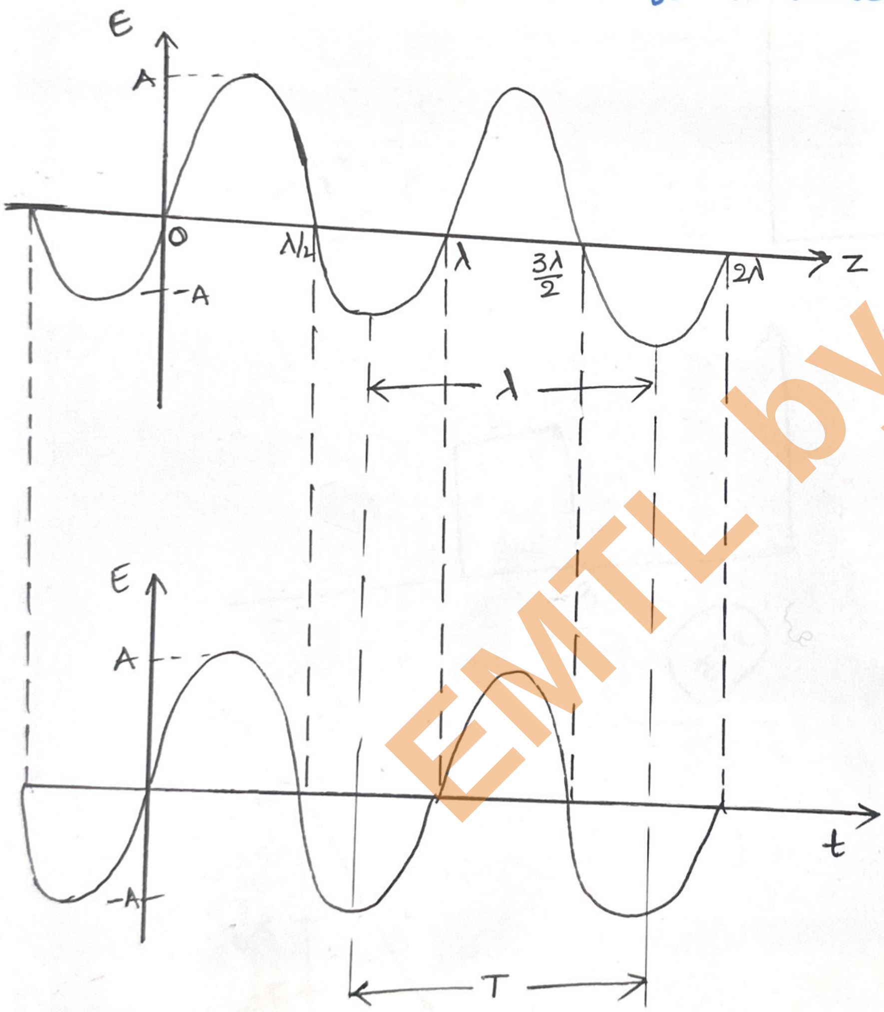


fig: Plot of $E(x,t) = A \sin(\omega t - \beta x)$

(a) with constant 't'

(b) with constant 'x'

from fig(a) observe that the wave takes distance 'λ' to repeat it self and hence 'λ' is called the wave length (in meters).

from fig(b) the wave takes time T to repeat itself. T is known as the period (in sec). Since it takes time T to travel distance λ at the speed 'v' i.e $\lambda = vT$

Velocity = d/t

$$(\text{or}) \quad v = \lambda/T = \lambda f$$

W.K.T $\quad v = \frac{\omega}{\beta} \Rightarrow \beta = \frac{1}{\frac{v}{\omega}} = \frac{\omega}{v} = \frac{2\pi f}{\lambda f}$

$$\therefore \boxed{\beta = \frac{2\pi}{\lambda}}$$

$\beta = \omega\sqrt{\mu\varepsilon}$

and also $\quad v = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu\varepsilon}}$

In free space $\quad v = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = \frac{1}{\sqrt{4\pi\times10^{-7}\times\frac{1}{36\pi\times10^{9}}}}$

$$c = v = 3\times10^{8}\ m/s$$

Wave propagation in a conducting medium:-

For a conducting medium $\quad \nabla^2 E = \mu\varepsilon\ddot{E} + \mu\sigma\dot{E}$

In phasor form $\quad \nabla^2 E = [\mu\varepsilon(j\omega)(j\omega) + \mu\sigma j\omega]E$

$$= (j\omega\mu\sigma - \omega^2\mu\varepsilon)E$$

$$\boxed{\nabla^2 E = \gamma^2 E} \quad \text{—— } ①$$

where $\gamma^2 = j\omega\mu\sigma - \omega^2\mu\varepsilon$, '$\gamma$' is called as Propagation constant.

The propagation constant γ is a complex number having real and imaginary parts designated by α and β respectively i.e $\gamma = \alpha + j\beta$

The uniform plane wave travelling in the x-direction the eqn ① can be written as $\quad \frac{d^2E}{dx^2} = \gamma^2 E$

$E(x) = E_0 e^{-\gamma x} + E_0 e^{\gamma x}$

but we are considering the wave along the +ve 'x' direction only.

which has only one possible solution $\quad E(x) = E_0 e^{-\gamma x}$

In time varying field $\quad \tilde{E}(x,t) = Re\left\{E_0 e^{-\gamma x + j\omega t}\right\}$

$$= Re\left\{E_0 e^{-(\alpha+j\beta)x + j\omega t}\right\}$$

$$= Re\left\{E_0 e^{-\alpha x + j(\omega t - \beta x)}\right\}$$

$$= e^{-\alpha x} Re\left\{E_0 e^{j(\omega t - \beta x)}\right\}$$

This is the equation of a wave travelling in the x-direction and attenuated by the factor $e^{-\alpha x}$.

α is called the attenuation constant (decibel/m (or) Nepers/m)

*** 1 db $\cong$ 0.115 NP = 0.115 nepers (or) 1 Neper = $\frac{20}{\ln 10}$ = 8.686 dB

consider $\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon) = (\alpha + j\beta)^2$

$$j\omega\mu\sigma - \omega^2\mu\varepsilon = \alpha^2 - \beta^2 + 2j\alpha\beta$$

Real part $\alpha^2 - \beta^2 = -\omega^2\mu\varepsilon$; $\beta = \frac{\omega\mu\sigma}{2\alpha}$

Substitute β in $\alpha^2 - \beta^2 = -\omega^2\mu\varepsilon$

$$\alpha^2 - \left(\frac{\omega\mu\sigma}{2\alpha}\right)^2 = -\omega^2\mu\varepsilon$$

$$\alpha^2 - \frac{\omega^2\mu^2\sigma^2}{4\alpha^2} = -\omega^2\mu\varepsilon$$

$$4\alpha^4 - \omega^2\mu^2\sigma^2 = -4\alpha^2\omega^2\mu\varepsilon$$

$$4\alpha^4 + 4\alpha^2\omega^2\mu\varepsilon = \omega^2\mu^2\sigma^2$$

$$\alpha^4 + \alpha^2\omega^2\mu\varepsilon = \frac{\omega^2\mu^2\sigma^2}{4} = \left[\frac{\omega\mu\sigma}{2}\right]^2$$

Add $\left[\frac{\omega^2\mu\varepsilon}{2}\right]^2$ both sides

$$\alpha^4 + \alpha^2\omega^2\mu\varepsilon + \left[\frac{\omega^2\mu\varepsilon}{2}\right]^2 = \left[\frac{\omega\mu\sigma}{2}\right]^2 + \left[\frac{\omega^2\mu\varepsilon}{2}\right]^2$$

$$\left(\alpha^2 + \frac{\omega^2\mu\varepsilon}{2}\right)^2 = \left[\frac{\omega^2\mu\varepsilon}{2}\right]^2\left[1 + \frac{\omega^2\mu^2\sigma^2}{4} \times \frac{4}{\omega^4\mu^2\varepsilon^2}\right]$$

$$= \left[\frac{\omega^2\mu\varepsilon}{2}\right]^2\left[1 + \frac{\cancel{\mu^2}\cancel{\omega^2}\sigma^2}{\omega^4\cancel{\mu^2}\varepsilon^2}\right]$$

$$= \left[\frac{\omega^2\mu\varepsilon}{2}\right]^2\left[1 + \frac{\sigma^2}{\omega^2\varepsilon^2}\right]$$

$$\left[\alpha^2 + \frac{\omega^2\mu\varepsilon}{2}\right] = \frac{\omega^2\mu\varepsilon}{2}\sqrt{1 + \frac{\sigma^2}{\omega^2\varepsilon^2}}$$

$$\alpha^2 = \frac{\omega^2\mu\varepsilon}{2}\sqrt{1 + \frac{\sigma^2}{\omega^2\varepsilon^2}} - \frac{\omega^2\mu\varepsilon}{2}$$

$$\alpha^2 = \frac{\omega^2\mu\varepsilon}{2}\left[\sqrt{1 + \frac{\sigma^2}{\omega^2\varepsilon^2}} - 1\right]$$

*** $$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \frac{\sigma^2}{\omega^2\varepsilon^2}} - 1\right]} \quad \text{Nepers/m}$$

III'ly ***

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \frac{\sigma^2}{\omega^2\varepsilon^2}} + 1\right]} \quad \text{radians/m}$$

1. Proove that the intrinsic impedance of a medium is given by

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$$

Sol: consider $\frac{\partial^2 E}{\partial z^2} = \gamma^2 E$ (or) uniform plane wave travelling in the z-direction

$$\therefore \frac{\partial^2 E}{\partial z^2} = \gamma^2 E \quad \& \quad E_z = H_z = 0$$

Which has one possible solution $E(z) = E_0 e^{-\gamma z}$ — ①

In time varying field $\vec{E}(z,t) = Re\{E_0 e^{-\gamma z} \cdot e^{j\omega t}\}$

from Maxwell's 2nd eqn

$$\nabla \times E = \dot{B} \Rightarrow \nabla \times E = -\mu \dot{H}$$

The time varying fields is represented in phasor form is

$$\nabla \times E = -j\omega\mu H$$

$\frac{\partial^2}{\partial z^2}(E_x \vec{x} + E_y \vec{y} + E_z \vec{z}) = \gamma^2(E_x \vec{x} + E_y \vec{y} + E_z \vec{z})$

comparing $\vec{x}$ terms

$\frac{d^2}{dz^2} E_x = \gamma^2 E_x$

$E_x = E_0 e^{-\gamma z}$

$$\begin{vmatrix} \vec{x} & \vec{y} & \vec{z} \\ 0 & 0 & \frac{d}{dz} \\ E_x & E_y & 0 \end{vmatrix} = -j\omega\mu [H_x \vec{x} + H_y \vec{y}]$$

$$\vec{x}\left(\frac{-\partial E_y}{\partial z}\right) + \vec{y}\left(\frac{\partial E_x}{\partial z}\right) = -j\omega\mu[H_x \vec{x} + H_y \vec{y}]$$

comparing $\vec{y}$ component

$$\frac{\partial E_x}{\partial z} = -j\omega\mu H_y$$

eqn ① Now Substitute $E_x = E_0 e^{-\gamma z}$

$$\frac{d}{dz}(E_0 e^{-\gamma z}) = -j\omega\mu H_y$$

$$-\gamma \times E_0 e^{-\gamma z} = -j\omega\mu H_y \Rightarrow \gamma E_x = j\omega\mu H_y$$

$$\frac{E_x}{H_y} = \frac{j\omega\mu}{\gamma} = \frac{j\omega\mu}{\sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}}$$

$$\frac{E_x}{H_y} = \sqrt{\frac{(j\omega\mu)^2}{j\omega\mu(\sigma + j\omega\varepsilon)}}$$

***

$$\boxed{\eta = \frac{E_x}{H_y} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}}$$

conductors and dielectrics :-

→ The Maxwell's first equation for sinusoidally time varying fields in phasor form can be expressed as

$$\nabla \times H = J + \frac{\partial D}{\partial t} = J_c + J_D = \sigma E + j\omega D$$

Where $J_c = \sigma E$; $J_D = j\omega D = j\omega\varepsilon E$

Further, the ratio of conduction current density to Displacement current density is given by $\left|\frac{J_c}{J_D}\right| = \left|\frac{\sigma E}{j\omega\varepsilon E}\right| = \left|\frac{\sigma}{j\omega\varepsilon}\right| = \frac{\sigma}{|j|\omega\varepsilon}$

$$= \frac{\sigma}{\omega\varepsilon} \qquad |j| = 1$$

***

Therefore $\left|\frac{J_c}{J_D}\right| = \frac{\sigma}{\omega\varepsilon} = 1$ can be considered to mark the dividing line between conductors and dielectrics. we may arbitrarily define '3' conditions as follows.

1. $\frac{\sigma}{\omega\varepsilon} << 1$ leads to $\omega\varepsilon >> \sigma$ i.e good dielectrics.
2. $\frac{\sigma}{\omega\varepsilon} = 1$ leads to $\omega\varepsilon \cong \sigma$ i.e quasi conductors.
3. $\frac{\sigma}{\omega\varepsilon} >> 1$ leads to $\omega\varepsilon << \sigma$ i.e good conductors

Wave propogation in good dielectrics :-

For this case $\frac{\sigma}{\omega\varepsilon} << 1$

$$\sqrt{1 + \frac{\sigma^2}{\omega^2\varepsilon^2}} = \left(1 + \frac{\sigma^2}{\omega^2\varepsilon^2}\right)^{1/2}$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots$$

$$\simeq \left(1 + \frac{1}{2}\frac{\sigma^2}{\omega^2\varepsilon^2}\right)$$

W.K.T $\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon) = (\alpha + j\beta)^2$

* The attenuation factor :

$$\alpha = \omega \sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}} - 1\right]}$$

$$= \omega \sqrt{\frac{\mu\varepsilon}{2}\left[\cancel{1}+\frac{\sigma^2}{2\omega^2\varepsilon^2} \cancel{-1}\right]}$$

$$= \sqrt{\frac{\omega^2\mu\varepsilon\sigma^2}{4\omega^2\varepsilon^2}} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$$

* Phase constant :

$$\beta = \omega \sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}} + 1\right]}$$

$$= \omega \sqrt{\frac{\mu\varepsilon}{2}\left[1+\frac{\sigma^2}{2\omega^2\varepsilon^2} + 1\right]}$$

$$= \omega \sqrt{\frac{\mu\varepsilon}{2}\left[\frac{\sigma^2}{2\omega^2\varepsilon^2} + 2\right]}$$

$$= \omega\sqrt{\mu\varepsilon}\sqrt{\frac{1}{2}\times 2\left[1+\frac{\sigma^2}{4\omega^2\varepsilon^2}\right]}$$

$$= \omega\sqrt{\mu\varepsilon}\sqrt{1+\frac{\sigma^2}{4\omega^2\varepsilon^2}} = \omega\sqrt{\mu\varepsilon}\left[1+\frac{\sigma^2}{8\omega^2\varepsilon^2}\right]$$

* Phase Velocity :

$$v = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{\mu\varepsilon}\left[1+\frac{\sigma^2}{8\omega^2\varepsilon^2}\right]}$$

$$\left[1+\frac{\sigma^2}{8\omega^2\varepsilon^2}\right]^{-1} = 1 - \frac{\sigma^2}{8\omega^2\varepsilon^2}$$

$$= \frac{1}{\sqrt{\mu\varepsilon}}\left[1-\frac{\sigma^2}{8\omega^2\varepsilon^2}\right]$$

* In freespace $v_0 = \frac{1}{\sqrt{\mu_0\varepsilon_0}} \Rightarrow v \cong v_0\left(1-\frac{\sigma^2}{8\omega^2\varepsilon^2}\right)$

* Intrinsic impedance $\eta = \sqrt{\frac{j\omega\mu}{\sigma+j\omega\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}\, j\omega\left[\frac{1}{j\omega}\right]\left[\frac{1}{1+\frac{\sigma}{j\omega\varepsilon}}\right]}$

$$\cong \sqrt{\frac{\mu}{\varepsilon}}\left(1+j\frac{\sigma}{2\omega\varepsilon}\right)$$

$$\frac{1}{\left(1+\frac{\sigma}{j\omega\varepsilon}\right)^{1/2}} = \left(1-\frac{\sigma}{2j\omega\varepsilon}\right)$$

$\downarrow$ (or)

$$\left(1+\frac{\sigma}{j\omega\varepsilon}\right)^{-1/2} = \left(1-\frac{\sigma j}{2j^2\omega\varepsilon}\right) = \left[1+\frac{j\sigma}{2\omega\varepsilon}\right]$$

EMTL by NKJ

Wave Propagation in a Good conductor :

For this case $\frac{\sigma}{\omega\varepsilon} \gg 1$

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} = \sqrt{j\omega\mu\sigma\left(1 + j\frac{\omega\varepsilon}{\sigma}\right)}$$

$\sigma \xrightarrow{\Xi} \infty$

$\rightarrow (1+0)$

Since conduction current is very large as compared to displacement current

$$\gamma \cong \sqrt{j\omega\mu\sigma} = \sqrt{\omega\mu\sigma}\,\angle 45^\circ$$

$$\gamma = \sqrt{\omega\mu\sigma}\,(\cos 45^\circ + j\sin 45^\circ) = \alpha + j\beta$$

$$\Rightarrow \sqrt{\frac{\omega\mu\sigma}{2}} + j\sqrt{\frac{\omega\mu\sigma}{2}} = \alpha + j\beta$$

Therefore $\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}}$

The Velocity of the wave in the conductor will be

$$v = \frac{\omega}{\beta} = \sqrt{\frac{\omega^2 \times 2}{\omega\mu\sigma}} = \sqrt{\frac{2\omega}{\mu\sigma}}$$

Intrinsic impedance of the conductor is

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = \sqrt{\frac{j\omega\mu}{\sigma\left(1 + \frac{j\omega\varepsilon}{\sigma}\right)}}$$

$$\cong \sqrt{\frac{j\omega\mu}{\sigma}} = \sqrt{\frac{\omega\mu}{\sigma}}\,\angle 45^\circ$$

$\theta = 90^\circ$

$\sqrt{j} = 45^\circ$

Let $\sqrt{i} = a + ib$

(or) $i = (a+ib)^2$

Backside $i = a^2 - b^2 + 2iab$

$a^2 - b^2 = 0$

$2iab = i$

$2ab = 1$

$ab = 1/2$

$a = 1/2b$

$\frac{1}{(2b)^2} - b^2 = 0$

$b^2 = \frac{1}{(2b)^2}$

$b^4 = \frac{1}{4}$

$b^2 = \frac{1}{2}$

$b = \frac{1}{\sqrt{2}} = a$

$\theta = \tan^{-1}(b/a)$

$= \tan^{-1}(1)$

$= \tan^{-1}(\tan 45^\circ)$

$= 45^\circ$

$\therefore \sqrt{i} = 45^\circ$

$\sqrt{j} = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$

$\frac{1}{\sqrt{2}}(1+j)$

Note :- It is seen that in good conductors where 'σ' is very Large, both α and β are also large. This means that the wave is attenuated greatly as it travels through the conductor and the phase shift per unit length is also Large.

* The characteristic impedance is also very small and has a reactive component. The angle of impedance is always 45° for good conductors.

* The velocity of the wave $(\propto \frac{1}{\beta})$ is very small in a good conductor.

Depth of Penetration (or) Skin depth :-

W.K.T $\quad \frac{d^2E}{dx^2} = \gamma^2 E$ $\qquad$ σ units - mho/meter (s)

the possible solution is $\quad E(x) = E_0 e^{-\gamma x}$

In phasor notation $\quad \bar{E}(x,t) = Re\{E_0 e^{-\gamma x} \cdot e^{j\omega t}\}$

$$= Re\{E_0 e^{-\alpha x} \cdot e^{-j\beta x} \cdot e^{j\omega t}\}$$

$$= Re\{E_0 e^{-\alpha x} \cdot e^{j(\omega t - \beta x)}\}$$

$= Re\{E_0 e^{-\alpha x} \cdot \cos(\omega t - \beta x) + j\sin(\omega t - \beta x)]\}$ $\quad = E_0 e^{-\alpha x}\cos(\omega t - \beta x) a_x$ we are taking the real part only

In good conductors $\quad \eta = \sqrt{\frac{\omega\mu}{\sigma}} \angle 45° = \frac{E}{H} = \frac{E_x}{H_y}$

$\sqrt{i} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}(1+i)$

$E_x = \dfrac{Re\{E_0 e^{-\alpha x} \cdot e^{j(\omega t - \beta x)}\}}{\sqrt{\frac{\omega\mu}{\sigma}} \angle 45°}$

$$H_y = \frac{E_x}{\eta} = \frac{E_0 e^{-\alpha x}\cos(\omega t - \beta x)}{\sqrt{\frac{\omega\mu}{\sigma}} \angle 45°}$$

$\dfrac{E_0 e^{-\alpha x} \cdot e^{j(\omega t - \beta x)}}{\sqrt{\frac{\omega\mu}{\sigma}}} \angle -45°$ $\downarrow \times e^{-j45°}$

$e^{-\alpha x} \cdot e^{j(\omega t - \beta x - 45°)}$

$\cos(\omega t - \beta x - 45°) + j\sin(\omega t - \beta x - 45°)$

Taking the real part

$$= \frac{E_0}{\sqrt{\frac{\omega\mu}{\sigma}}} e^{-\alpha x}\cos(\omega t - \beta x - 45°) a_y$$

$\angle -45° = \frac{1}{\sqrt{2}}(1-i) = \cos 45° - i\sin 45°$, i.e. $= e^{-i45°}$

Therefore, as E or H wave travels in a conducting medium, its amplitude is attenuated by the factor $e^{-\alpha x}$. The distance 'δ' as shown in fig: through which the wave amplitude decreases by a factor e^{-1} (about 37%) is called skin depth or depth of Penetration. denoted by 'δ'

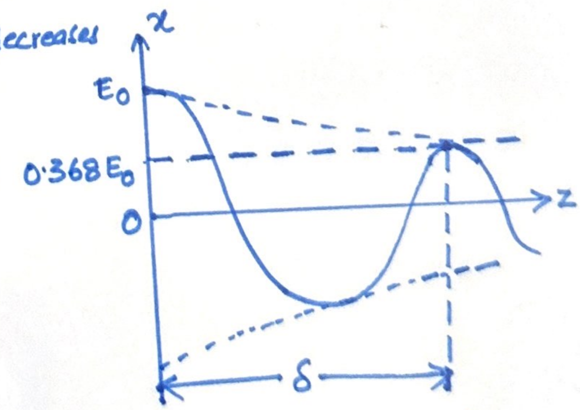


fig: Illustration of skin depth.

$$E_0 e^{-\alpha x} = E_0 e^{-1}$$

distance $x = \alpha$

$$E_0 e^{-\alpha\delta} = E_0 e^{-1}$$

$$\alpha\delta = 1 \Rightarrow \delta = \frac{1}{\alpha}$$

Skin depth is a measure of the depth to which an EM wave can penetrate the medium.

How much distance the wave travels in a good conducting medium that distance is called skindepth (or) depth of penetration

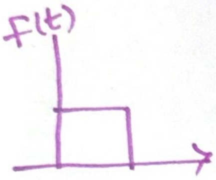
Wave propagation in a conducting medium

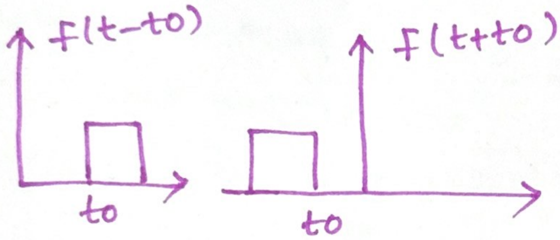
$$\frac{\partial^2 E}{\partial x^2} = \gamma^2 E \rightarrow D^2 - \gamma^2 = 0$$

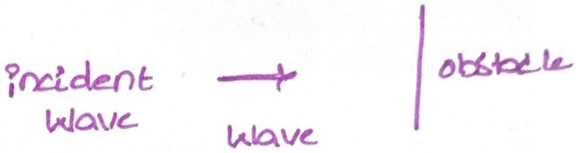
$$D = \pm \gamma$$

The possible solution is

$$E(x) = E_0 e^{-\gamma x} + E_0 e^{\gamma x}$$







wave travelling in the $+x$ direction. There is no obstacle there is no reflection (or the complete wave is (travelling) transmitted.

$\therefore E_0 e^{\gamma x} = 0$

$-x$ is in forward direction (or (+ve axis) incident wave

$+x$ is in reverse direction (or reflected wave. (-ve)

***

$E_0 e^{-\alpha x}$ $\alpha = 1$ $x = 0, 1, 2, \ldots \infty$

because of the amplitude decreases from E_0 to ..... '∞'

that's why 'α' is called as attenuation constant

***

The ratio of magnitudes of conduction current density and displacement current density is $\left|\frac{\sigma E}{j\omega \epsilon E}\right| = \frac{\sigma}{\omega \epsilon}$ which serves a boundary between conductors & dielectrics and this ratio is called Loss tangent.

↳ time varying form

$\frac{dD}{dt} = j\omega D = j\omega \epsilon E$

For good conductors $\frac{\sigma}{\omega \epsilon} >> 1$

For good dielectrics $\frac{\sigma}{\omega \epsilon} << 1$.

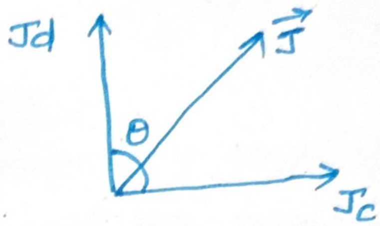
For Perfect dielectrics the ratio tends to zero.

$\frac{\sigma}{\omega\varepsilon} = = 1$ for good semiconductors.

$$\tan\theta = \frac{J_d}{J_c} = \frac{\sigma}{\omega\varepsilon}$$

$\tan\theta \downarrow$ Loss tangent

$\frac{\sigma}{\omega\varepsilon} \uparrow$ dissipation factor



Skin depth :

Depth of Penetration : In a medium which has conductivity, the wave is attenuated as it progresses, owing the losses which occur. At high frequencies, in good conductors, the rate of attenuation is high and the way it penetrates only a short distance before it reduces to a negligibly small value.

Depth of penetration δ is defined as the depth in which the wave has been attenuated to $1/e$ (on approximately 37% of its original value.

→ since amplitude decreases by a factor $e^{-\alpha x}$, for the amplitude to fall to $1/e$ of its original value.

$$E(x,t) = e^{-\alpha x}\left\{Re\ E_0 e^{j(\omega t-\beta x)}\right\}$$

$$\frac{E_0}{e} = E_0 e^{-\alpha x}$$

$x = \delta \Rightarrow e^{-1} = e^{-\alpha\delta} \Rightarrow \alpha\delta = 1$

$$\boxed{\delta = 1/\alpha}$$

$$\delta = \frac{1}{\omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}}-1\right]}}$$

for good conductors $\delta = \frac{\sqrt{2}}{\sqrt{\omega\mu\sigma}}$

for perfect conductor $\sigma \to \infty \Rightarrow \delta = 0$

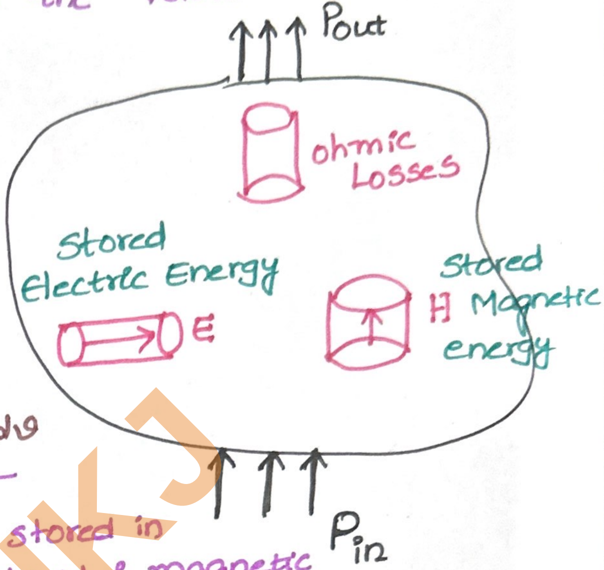
For a good conductor the depth of Penetration is

$$\boxed{\delta = \frac{1}{\alpha} \cong \sqrt{\frac{2}{\omega\mu\sigma}}}$$

* The depth of Penetration may be defined as that depth in which the wave has been attenuated by amount (1/e) on 37% approximately of its initial value.

Poynting theorem :-

states that the net power flowing out of a given volume which is free of sources is equal to the time rate of decrease of electric and magnetic energy stored in the volume minus (-) conduction Losses (ohmic Losses)



$$\int_v E \cdot J\, dv = -\frac{d}{dt}\int_v \left(\frac{1}{2}\mu H^2 + \frac{1}{2}\varepsilon E^2\right) dv - \oint_s (E \times H)\, ds$$

$$\underbrace{\oint_s (E\times H)\, ds}_{\text{Power flowing out of the volume}} = \underbrace{-\frac{d}{dt}\int_v \left(\frac{1}{2}\mu H^2 + \frac{1}{2}\varepsilon E^2\right) dv}_{\text{Energy stored in electric field \& magnetic field}} \underbrace{- \int_v E \cdot J\, dv}_{\text{conduction Losses}}$$

$$P_{out} = \oint_s (E \times H)\, ds$$

$E \times H = P =$ Poynting vector

$\frac{A}{m} \cdot \frac{V}{m} = W/m^2$ so $P =$ Power density

① E - normal to the plane of incidence
H in the plane of incidence
→ Perpendicular (or) Horizontal polarization

② E - in the plane of incidence
H normal to the plane of incidence
→ Parallel (or) Vertical polarization

Problems :

1> A plane sinusoidal electromagnetic wave travelling in freespace has $E_{max} = 1500\ \mu V/m$. find the accompanying H_{max}.

Sol : Given that $E_{max} = 1500 \mu V/m$

The electromagnetic wave is travelling in free space ($\mu_r = \varepsilon_r = 1$)

$$\varepsilon = \varepsilon_0 , \quad \mu = \mu_0 ; \quad \sigma = 0$$

and $\eta_0 = \sqrt{\dfrac{\mu_0}{\varepsilon_0}} = 377\Omega$

W.K.T $\dfrac{E_x}{H_y} = \eta \Rightarrow \eta = \dfrac{1500 \times 10^{-6}}{H_y} \Rightarrow H_y = \dfrac{1500 \times 10^{-6}}{377}$

$$= 3.98\ \mu A/m$$

2> A uniform plane wave at a frequency of 1 GHz is travelling in a large block of Teflon ($\varepsilon_r \approx 2.1$, $\mu_r = 1$ and $\sigma = 0$). determine v_p, η, β and λ

Sol : Given that $\varepsilon_r = 2.1$, $\mu_r = 1$, $\sigma = 0 \rightarrow$ represents free space

$f = 1 \times 10^9$ (Gega) Hz

i> $\beta = \omega\sqrt{\mu\varepsilon} = 2\pi f \sqrt{\mu_0 \varepsilon_0 \times \mu_r \varepsilon_r}$

$$= 2\pi \times 10^9 \sqrt{4\pi \times 10^{-7} \times \frac{1}{\frac{36\pi \times 10^9}{9}} \times 2.1}$$

$$= \underline{30.3\ \text{rad/m}}$$

ii) $v_p = \dfrac{\omega}{\beta} = \dfrac{2\pi f}{\beta} = \dfrac{2\pi \times 10^9}{30.3} = \underline{2 \times 10^8\ \text{m/sec}}$

iii) $\lambda = \dfrac{2\pi}{\beta} = \dfrac{2\pi}{30.3} = \underline{0.2\text{m}}$

iv) $\eta = \sqrt{\dfrac{\mu}{\varepsilon}} = \sqrt{\dfrac{\mu_0 \mu_r}{\varepsilon_0 \varepsilon_r}} = \sqrt{\dfrac{4\pi \times 10^{-7} \times 1}{8.854 \times 10^{-12} \times 2.1}}$

$$= \underline{260\Omega}$$

37> A 100 V/m plane wave of freq 300 MHz travels in an infinite lossless medium having $\mu_r = 1$, $\varepsilon_r = 9$, $\sigma = 0$. Write the complete time domain equations for E and H field vectors

Sol: Given that $E_0 = 100\ V/m$; $f = 300 \times 10^6$ Hz

$\mu_r = 1$, $\varepsilon_r = 9$; $\sigma = 0$

For a loss less medium $\alpha = 0$ $\alpha = \omega \sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \frac{\sigma^2}{\omega^2\varepsilon^2}} - 1\right]}$ neper/m

when $\sigma = 0 \Rightarrow \alpha = 0$

$$\beta = \sqrt{\frac{\omega^2\mu\varepsilon}{2}\left[\sqrt{1 + \frac{\sigma^2}{\omega^2\varepsilon^2}} + 1\right]} = \omega\sqrt{\mu\varepsilon}$$

$$= 2\pi \times 300 \times 10^6 \times \sqrt{4\pi \times 10^{-7} \times 1 \times \frac{1}{36\pi \times 10^9} \times 9}$$

$$= 18$$

uniform plane wave travelling in the z-direction is

$$E = E_0 \cos(\omega t - \beta z)\, a_x \times e^{-\alpha z}$$

$$= 100 \cos(2\pi \times 300 \times 10^6 t - 18z) \times e^{-0} a_x$$

$$= 100 \cos(1.8 \times 10^9 t - 18z)\, a_x$$

$$\frac{E_x}{H_y} = \eta \Rightarrow H_y = \frac{E_x}{\eta} = \frac{100 \cos(1.8 \times 10^{-9} t - 18z)}{\sqrt{\frac{\mu_0\mu_r}{\varepsilon_0\varepsilon_r}}}$$

$$\eta = \sqrt{\frac{4\pi \times 10^{-7} \times 1}{8.854 \times 10^{-12} \times 9}} = 125 \qquad = \frac{100 \cos(1.8 \times 10^{-9} t - 18z)}{125}$$

$$\therefore H_y = 0.8 \cos(1.8 \times 10^{-9} t - 18z)\, a_y$$

47 Determine the phase velocity of propagation, attenuation constant, phase constant and intrinsic impedance for a forward travelling wave in a large block of copper at 1 MHz ($\sigma = 5.8\times10^{7}$, $\varepsilon_r = \mu_r = 1$) . determine the distance that the wave must travel to be attenuated by a factor of 100 (40 dB)

Sol: $f = 1MHz = 1\times10^{6}\,Hz$; $\sigma = 5.8\times10^{7}$, $\varepsilon_r = \mu_r = 1$ calculate

V_p, α, β and η

wave travelling in a large block of copper at (1MHz , $\sigma = 5.8\times10^{7}$)

$$\frac{\sigma}{\omega\varepsilon} = \frac{5.8\times10^{7}}{2\pi\times10^{6}\times8.854\times10^{-12}}$$

$$= \frac{5.8\times10^{7}}{2\pi\times8.854\times10^{-6}}$$

$$\frac{\sigma}{\omega\varepsilon} >> 1$$

Therefore wave travelling in a good conducting medium.

i) $$V_p = \frac{\omega}{\beta} = \frac{\omega}{\sqrt{\frac{\omega\mu\sigma}{2}}} = \sqrt{\frac{2\omega}{\mu\sigma}}$$

$$= \sqrt{\frac{2\times2\times\pi\times10^{6}}{4\pi\times10^{-7}\times5.8\times10^{7}}} = 415.22\ m/sec$$

ii) $$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\frac{2\times\pi\times10^{6}\times4\pi\times10^{-7}\times5.8\times10^{7}}{2}}$$

$$\alpha = 15131.9\ Np/m$$

$$\beta = 15131.9\ radians/m$$

iii) $$\eta = \sqrt{\frac{j\omega\mu}{\sigma}} = \angle 45^\circ \times \frac{\omega\mu}{\sigma} = \sqrt{\frac{2\times\pi\times10^{6}\times4\pi\times10^{-7}}{5.8\times10^{7}}}\ \angle 45^\circ$$

$$= 3.68\times10^{-4}\ \angle 45^\circ$$

iv) $$\text{distance} : \delta = \sqrt{\frac{2}{\omega\mu\sigma}}$$

$$= \sqrt{\frac{2}{2\pi\times10^{6}\times4\pi\times10^{-7}\times5.8\times10^{7}}}$$

$$= 6.61\times10^{-5}\ m$$

57) Find μ_r, ε_r and σ for a material in which at 100 MHz, uniform plane wave has $\alpha = 2$ Np/m, $\lambda = 1$m and $|\eta| = 200\Omega$

Sol: $\alpha = 2$ Np/m ; $\eta = 200\Omega$; $\lambda = 1m$; $f = 100$ MHz

$$\boxed{\lambda = \frac{2\pi}{\beta}}$$

$$V = \frac{\omega}{\beta} = \frac{2\pi f}{\beta} = \frac{\omega}{\omega\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu\varepsilon}} \quad - (1)$$

$$V = \frac{2\pi f}{2\pi} \times \lambda = f \times \lambda = 100 \times 10^6 \times 1 = 1 \times 10^8 \quad - (2)$$

Equating (1) & (2)

$$\frac{1}{\sqrt{\mu_0\varepsilon_0\mu_r\varepsilon_r}} = 10^8 \Rightarrow \frac{1}{\sqrt{\mu_r\varepsilon_r}} = 10^8 \times \sqrt{4\pi \times 10^{-7} \times 8.854 \times 10^{-12}}$$

$$= 10^8 \times 3.39 \times 10^{-9}$$

$$\sqrt{\mu_r\varepsilon_r} = 3 \quad \text{— eqn (3)}$$

$$\eta = \sqrt{\frac{\mu_0\mu_r}{\varepsilon_0\varepsilon_r}} = \sqrt{\frac{4\pi \times 10^{-7}}{8.854 \times 10^{-12}} \times \frac{\mu_r}{\varepsilon_r}}$$

$$200 = 377\sqrt{\frac{\mu_r}{\varepsilon_r}} \Rightarrow \sqrt{\frac{\mu_r}{\varepsilon_r}} = 0.53$$

$$\underline{\sqrt{\mu_r} = \sqrt{\varepsilon_r}\, 0.53}$$

Substitute in eqn (3) we get

$$\sqrt{\mu_r} = \sqrt{\varepsilon_r}\, 0.53 \Rightarrow \sqrt{\varepsilon_r}\sqrt{\varepsilon_r}\, 0.53 = 3$$

$$\varepsilon_r = 3/0.53 = \underline{\underline{5.66}}$$

$$\sqrt{\mu_r} = \frac{3}{\sqrt{5.66}} \Rightarrow \mu_r = \underline{1.59}$$

$$\alpha \text{ in good conducting medium} = \sqrt{\frac{\omega\mu\sigma}{2}}$$

$$\alpha^2 = \frac{\omega\mu\sigma}{2} \Rightarrow 4 = \frac{2\pi \times 10^8 \times 4\pi \times 10^{-7} \times 1.59 \times \sigma}{2}$$

$$4 = 6.27 \times \sigma \Rightarrow \sigma = \frac{4}{6.27 \times 10^2}$$

$$= 0.637 \times 10^{-2}$$

$$= \underline{\underline{6.37 \times 10^{-3}}}$$

6> A c^{lar} Loop conductor of radius 0.1 m lies in the z=0 plane and has a resistance of 5Ω given $B = 0.20 \sin 10^3 t\, a_z$. determine the current.

Sol :
$$B = 0.20 \sin 10^3 t\, a_z$$

Radius of circular Loop conductor $\rho = 0.1\,m$

Resistance $R = 5\Omega$ $\qquad ds = \rho\, d\rho\, d\phi\, dz$

$$\phi = \iint_S \vec{B}\cdot\vec{ds} = \int_{\rho=0}^{0.1}\int_{\phi=0}^{2\pi} (0.20 \sin^3 t)\, a_z \cdot \rho\, d\rho\, d\phi\, a_z$$

$$= 0.20 \sin 10^3 t \times \left[\frac{\rho^2}{2}\right]_0^{0.10} \times 2\pi$$

$$\phi = \frac{2\pi \sin^3 10^3 t}{10^3} = \frac{2\pi \sin 10^3 t}{10^3}$$

The voltage induced in the conductor is obtained by differen-tiating the flux w.r.to time i.e

$$V = -\frac{d\phi}{dt} = -\frac{2\pi \cos 10^3 t \times 10^3}{10^3} = -2\pi \cos 10^3 t \text{ Volts.}$$

−ve sign indicates that when flux increases in positive direction the induced voltage is −ve.

From ohm's law => $V = IR \Rightarrow R = \frac{V}{I} \Rightarrow I = \frac{V}{R}$

$$I = \frac{2\pi \times \cos 10^3 t}{5} = 1.26 \cos 10^3 t \text{ Amp.}$$

7> A certain material has $\sigma = 0$ & $\varepsilon_r = 1$ if $H = 4\sin(10^6 t - 0.01z)\,\vec{a_y}$ A/m. Make use of Maxwell's eqn's to find μ_r.

Sol : Given that $\sigma = 0$, $\varepsilon_r = 1$ & $H = 4\sin(10^6 t - 0.01z)\vec{a_y}$

$$\nabla\times\vec{H} = \begin{vmatrix} \vec{a_x} & \vec{a_y} & \vec{a_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 4\sin(10^6 - 0.01z) & 0 \end{vmatrix}$$

$$\nabla\times\vec{H} = -\vec{a_x}\left(\frac{\partial}{\partial z}\left(4\sin(10^6 t - 0.01z)\right)\right)$$

$$= -\vec{a_x} \times 4 \times \cos(10^6 t - 0.01z) \times (-0.01)$$

$$= 0.04\cos(10^6 t - 0.01z)\,\vec{a_x}$$

$$\nabla\times\vec{H} = J + \frac{\partial D}{\partial t} \quad \text{when } \sigma = 0 \Rightarrow J = 0$$

$$\nabla\times H = \frac{\partial D}{\partial t} \Rightarrow \vec{D} = \frac{0.04}{10^6}\sin(10^6 t - 0.01z)\,\vec{a_x}$$

$$= 4\times 10^{-8}\sin(10^6 t - 0.01z)\,\vec{a_x}$$

$$D = \varepsilon E \Rightarrow E = \frac{4\times 10^{-8}\sin(10^6 t - 0.01z)\,\vec{a_x}}{\varepsilon_0} \qquad \boxed{\varepsilon_r = 1}$$

$$\nabla\times E = \begin{vmatrix} \vec{a_x} & \vec{a_y} & \vec{a_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{4\times 10^{-8}}{\varepsilon_0}\sin(10^6 t - 0.01z)\,\vec{a_x} & 0 & 0 \end{vmatrix}$$

$$= \vec{a_y}\left(\frac{\partial}{\partial z}\frac{4\times 10^{-8}}{\varepsilon_0}\sin(10^6 t - 0.01z)\right) \Big| \vec{a_y} = \frac{4\times 10^{-8}}{\varepsilon_0}\cos(10^6 t - 0.01z) \times -(0.01)$$

$$\Rightarrow -\frac{4\times 10^{-10}}{\varepsilon_0}\cos(10^6 t - 0.01z)$$

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \Rightarrow \vec{B} = \int \frac{4\times 10^{-10}}{\varepsilon_0}\cos(10^6 t - 0.01z)\,dt$$

$$\vec{B} = \vec{\mu}\vec{H} \qquad = \frac{4\times 10^{-16}}{\varepsilon_0}\sin(10^6 t - 0.01z)\,a_y$$

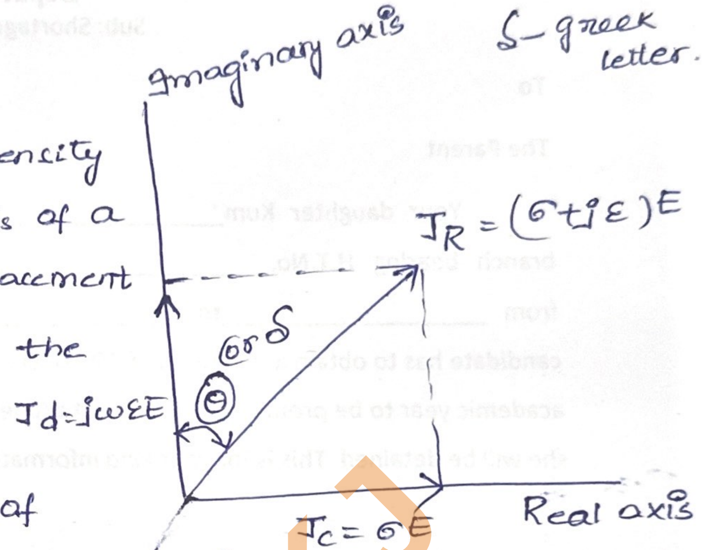
$$\vec{\mu} = \vec{B}/\vec{H} = \frac{10^{-16}}{\varepsilon_0} \Rightarrow \mu_r = 10^{-16}/\mu_0\varepsilon_0 = \underline{8.9875}$$

D.F → How much loss would result.

Loss tangent is defined as the tangent of the difference of the phase angle between J_c & J_D. this difference being caused by the losses in the (material) medium.

The Loss Tangent:

If the conduction current density is taken along the real axis of a complex plane, then the displacement current density will be along the imaginary axis as shown in Fig: Such a representation of sinusoidaly time varying quantities is called phasor diagram.



The resultant current density J_R is the phasor sum of the two current densities.

The time-phase difference J_d and J_R is denoted by θ. Accordingly

$$\text{Losstangent} = \tan\theta = \left|\frac{J_c}{J_d}\right| = \frac{\sigma}{\omega\varepsilon}$$

The value of tanθ is a measure of the magnitude of conduction current density relative to that of the displace current density.

A low value of tanθ for a material means that the material is of Low-Loss type. The attenuation constant of the material is also small. The ratio $\sigma/\omega\varepsilon$ is called loss tangent (or) dissipation factor.

*** ratio at any particular freq btwn the real & imaginary parts of the current densities

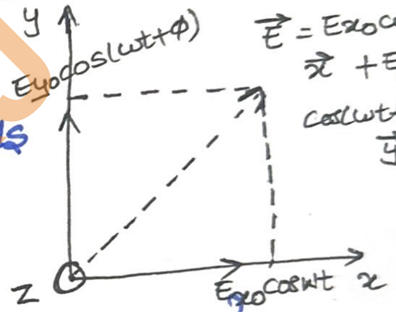
Polarization :-

→ The polarization of a uniform plane wave refers to the time-varying behaviour of the electric field strength vector at some fixed point in space.

→ The state of polarisation of a wave is described by the geometrical shape drawn by tip of the electric field vector as a function of time at a given point in space.

→ consider the uniform plane wave propagating in the +ve z-direction having electric fields along x and y directions respectively ($E_z = 0$)

→ Let the amplitudes of the two fields be different and Let there be a phase difference between them.



$\vec{E} = E_{xo}\cos\omega t\, \vec{x} + E_{yo}\cos(\omega t+\phi)\, \vec{y}$

→ Assuming the fields to be time-varying fields

$$E_x = E_{xo} e^{-j\beta z} \qquad E_y = E_{yo} e^{-j\beta z}$$

In time varying fields

$$\tilde{E}_x = Re\{E_{xo} e^{-j\beta z} \cdot e^{j\omega t}\} \qquad \tilde{E}_y = Re\{E_{yo} e^{-j\beta z} \cdot e^{j\omega t + \phi}\}$$

$$\tilde{E}_x = E_{xo}\cos(\omega t - \beta z) \qquad \tilde{E}_y = E_{yo}\cos(\omega t - \beta z + \phi)$$

$\phi \to$ phase difference

let $z=0$ (At a fixed point ($z=0$) say

$$\tilde{E}_x = E_{xo}\cos\omega t \; ; \quad \tilde{E}_y = E_{yo}\cos(\omega t + \phi)$$

$$\cos\omega t = \frac{E_x}{E_{xo}} \; ; \quad E_y = E_{yo}[\cos\omega t\cos\phi - \sin\omega t\sin\phi]$$

$$\sin\omega t = \left[1 - \left(\frac{E_x}{E_{xo}}\right)^2\right]^{1/2} \qquad = E_{yo}\left[\frac{E_x}{E_{xo}}\cos\phi - \left(1 - \frac{E_x^2}{E_{xo}^2}\right)^{1/2}\sin\phi\right]$$

$$\Rightarrow \left[\frac{E_y}{E_{yo}} - \frac{E_x}{E_{xo}}\cos\phi\right]^2 = \left[-\left(1 - \frac{E_x^2}{E_{xo}^2}\right)^{1/2}\sin\phi\right]^2$$

Squaring on both sides

$$\left(\frac{E_y}{E_{yo}}\right)^2 + \left(\frac{E_x}{E_{xo}}\right)^2\cos^2\phi = \left(1 - \frac{E_x^2}{E_{xo}^2}\right)\sin^2\phi$$

after expansion

$$\frac{E_y^2}{E_{yo}^2} + \frac{E_x^2}{E_{xo}^2} - 2\frac{E_x E_y\cos\phi}{E_{xo}E_{yo}} = \sin^2\phi \qquad \text{— (1)}$$

Equations :

→ Ellipse : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$a = b = 1$ ∴

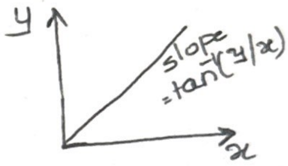
→ circle $x^2 + y^2 = 1$

→ $y = mx$ straight line.

The above eqn is the general form of ellipse

Observation :- The two Electric field components E_x & E_y having different amplitudes and are not in phase i.e if they reach their max values at different instants of time, then the direction of the resultant electric vector with vary with time. In this case it can be shown that the Locus of the end point of the resultant $\tilde{E}$ will be an ellipse and the wave is said to be elliptically polarized.

Case 1 :- The two components E_x & E_y may (or) may not have same amplitude but let us assume that the phase difference between them is zero.



Let $\phi = 0$ → the eqn ① is reduced to

$$\left[\frac{E_x}{E_{x0}} - \frac{E_y}{E_{y0}}\right]^2 = 0 \Rightarrow \frac{E_y}{E_{y0}} = \frac{E_x}{E_{x0}} \Rightarrow E_y = \left[\frac{E_{y0}}{E_{x0}}\right] E_x \quad \text{— (2)}$$

$$y = m x$$

The above eqn represents a straight line with slope $[E_{y0}/E_{x0}]$. If E_x & E_y are in same phase $(\phi = 0)$ the wave is said to be Linearly polarized and the tip of the electric field Vector traces a st. line irrespective of the amplitudes of the two fields components.

→ If $E_{x0} = 0 \Rightarrow m = \infty$ $\tan^{-1}(\tan 90°) = 90°$, then the line becomes vertical and the wave is Vertically polarized.

→ If $E_{y0} = 0 \Rightarrow m = 0$ $\tan^{-1}(\tan 0°) = 0°$, then the line becomes horizontal and the wave is horizontally polarized.

→ If $E_{x0} = E_{y0} \Rightarrow m = 1$ $\tan^{-1}(\tan 45°) = 45°$. the the wave is said to be linearly polarized with a polarizing angle of $45°$

see back side of the Page

Case 2 :- $\phi = \pm\pi/2$ and $E_{x0} = E_{y0} = E_0 \Rightarrow$ the eqn ① becomes

$$E_x^2 + E_y^2 = E_0^2$$ → This is the eqn of a circle.

If the y component leads the x- component by 90° and if both the x & y components of the electric fields are equal in magnitude then the wave is said to be circularly polarized. If $\phi = \oplus\pi/2$ → left circularly polarized.

$\phi = -\pi/2$ → It is right circularly polarized.

Questions from JNTU Exam Papers

1> a) State and Prove Gauss's law. List the limitations of Gauss' Law

b) Derive an expression for the electric field strength due to a circular ring of radius 'a' and uniform charge density ρ_L c/m using Gauss Law. obtain the value of height 'h' along z-axis.

c) Define electric potential.

2> a) State Maxwell's equations for Magnetostatic fields.

b) Show that the magnetic field due to finite current along z-axis at a point P, 'r' distance away along y-axis is given by $H = \frac{I}{4\pi}(\sin\alpha_1 - \sin\alpha_2)\hat{a}_\phi$ where I is the current through the conductor α_1 and α_2 are the angles made by the tips of the conductor element at P.

3> a) The electric field intensity in the region $0 < x < 5$, $0 < y < \pi/12$ and $0 < z < 0.06$ m in freespace is given by $E = C \sin 12y \sin az \cos(2 \times 10t)\, a_x$ V/m. Beginning with the $\nabla \times E$ relationships use Maxwell's equations to find a numerical value of a, if it is known that 'a' is greater than '0'

4> a) For good dielectrics derive the expressions for α, β, v and η

b) Express Gauss Law in both integral and differential forms

c) Derive poisson's and Laplace eqn's starting from Gauss Law

5> a) Write down the Maxwell's eqn's for Harmonically varying fields

b) For a conducting medium derive expressions for 'α' & 'β'

6> a) State and Prove Ampere's circuital law. specify the conditions to be met for determining magnetic field strength H, based on Ampere's circuit law.

b) Define magnetic flux density and vector magnetic potential

7. a> using Gauss' law derive expressions for electric field intensity and electric flux density due to an infinite sheet of conductor of charge density ρ c/m

b> A parallel plate capacitance has 500 mm side plates of square shape seperated by 10 mm distance. A sulphur slab of 6 mm thickness with $\varepsilon_r = 4$ is kept on the lower plate. find the capacitance of the set-up. If a voltage of 100 volts is applied across the capacitor, calculate the voltages at both the regions of the capacitor between the plates. $\left[C = \dfrac{\varepsilon_0 A}{\frac{d_1}{\varepsilon_{r1}} + \frac{d_2}{\varepsilon_{r2}}}\right]$

8. a> Derive equation of continuity for static magnetic fields?

b> What is inconsistency of Ampere's law?

9. a> Write down the Maxwell's eqn's for Harmonically varying fields?

b) Apply Gauss's law to derive the boundary conditions at a conductor - dielectric interface.

10. a> State and explain coulomb's law using vector form of coulomb? and force expression.

b> State and express Gauss's law in both integral and differential form?

c> In freespace $D = D_m \sin(\omega t + \beta z)\, a_x$ use maxwell's eqn's to find 'B'.

Including all basic definitions of TL & problems & basic Laws in the Notes.

11. What is meant by polarization? When it is elliptically, circularly and linearly polarized!

UNIT – V

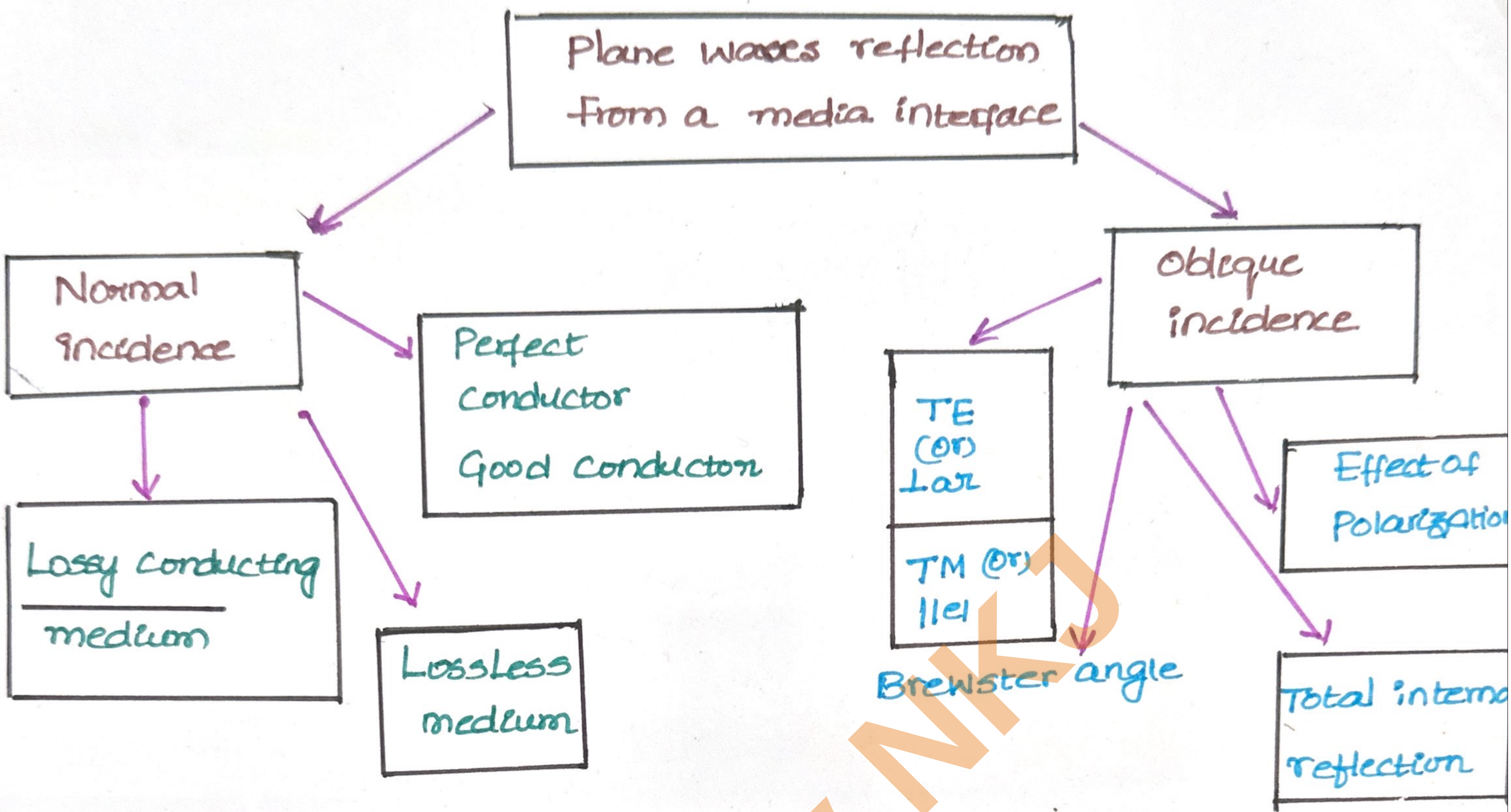
EM WAVE CHARECTERISTICS – II

UNIT – 5

References

1. Balman & Jordan

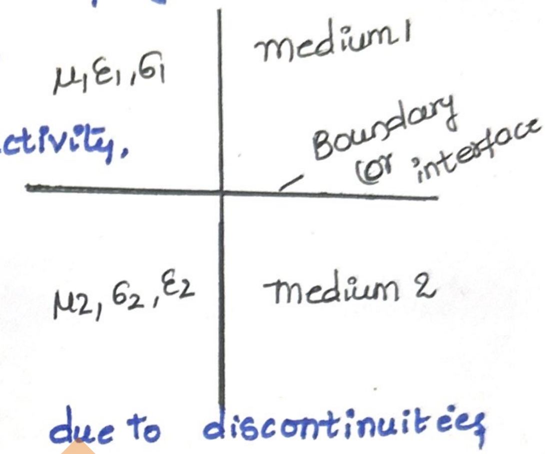
INTRODUCTION:-



* The ratio of reflected wave / incident wave is known as the reflection coefficient of the surface.
* This ratio depends on the σ, ε & μ of the material that forms the reflective surface
* The ratio of the transmitted wave divided by the incident wave is known as the transmission coefficient [How much of the incident wave has been transmitted through the medium (or) material]
* Apply the boundary conditions to get the values of the transmission & reflection coefficients.
* For smooth surfaces EM waves are reflected
  " Rough " " " " Scattered
  " edges of " " " " diffracted.
  ↓
  breaking up incoming wave by some sort of geometrical structure

Reflection by a Perfect conductor - Normal incidence

→ When an em wave travelling in one medium impinges (incident) upon a second medium having a different dielectric constants, Permeability (or) conductivity, the wave (in general) will be partially transmitted and partially reflected.



due to discontinuities at the interface, the wave is partially transmitted & ↓ reflected.

→ As the depth of penetration

$$\delta = \sqrt{\frac{2}{\omega \mu \sigma}}$$

For perfect conductor (good conductor)

$$\boxed{\sigma \cong \infty \quad \Rightarrow \quad \delta = 0}$$

media is freespace - conductor

∴ The wave in air incident normally upon the surface of a Perfect conductor, the wave is entirely reflected. because there is no absorption (or) transmission of em wave ($\delta = 0$) the entire wave is reflected back.

As a result the amplitudes of E and H in the reflected wave are the same as in the incident way and the only difference is in the direction of power flow (direction of propaga-tion gets reversed)

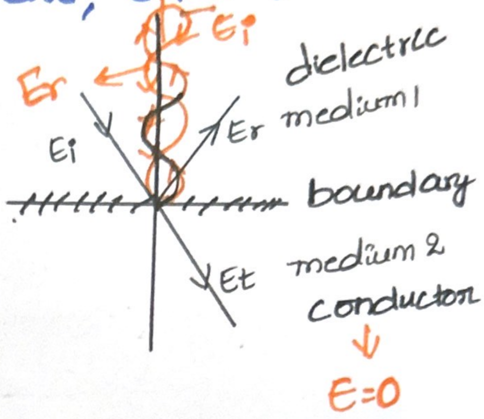
→ Neither E nor H can exist with in a perfect conductor.

→ The expressions for the electric field of the incident wave is $\boxed{E_i\, e^{-j\beta x}}$ (+ve x-direction)

→ The expression for the electric field of the reflected wave will be $\boxed{E_r\, e^{j\beta x}}$ (-ve x-direction)

From the Boundary conditions $\boxed{E_{1t} = E_{2t}}$

The tangential component of 'E' must be continueous across the boundary. Since the waves are tranverse, E & H fields are entirely tangential to the interface.



In medium 1 The total electric field

$$\boxed{E_1 = E_i + E_r}$$

In medium 2 $\boxed{E_2 = E_t}$

$\therefore \quad E_{1t} = E_{2t} \Rightarrow E_i + E_r = E_t$

III'ly for magnetic fields $\quad H_i + H_r = H_t$

For a perfect conductor $\quad E_t = 0$ ($\sigma \to \infty$ $\quad J = \sigma E$, $E = \frac{J}{\sigma} = 0$)

$$E_t = E_i + E_r$$

$$0 = E_i + E_r \Rightarrow \boxed{E_r = -E_i}$$ ***

The amplitude of the reflected electric field strength is equal to that of the incident electric field strength but its phase has been reversed.

→ The resultant electric field strength at any point is equal to the sum of the field strengths of incident & reflected waves.

$$\boxed{E_T(x) = E_i + E_r}$$

$$= E_i e^{-j\beta x} + E_r e^{j\beta x}$$

$$= E_i e^{-j\beta x} - E_i e^{j\beta x} \quad (E_r = -E_i)$$

$$= E_i \left[e^{-j\beta x} - e^{j\beta x} \right]$$

$$= -E_i [2j \sin\beta x]$$

In time varying fields

$$\tilde{E}_T(x,t) = Re\left\{ -E_i [2j \sin\beta x] e^{j\omega t} \right\}$$ Taking the real part

$$= -2j E_i \sin\beta x \times j \sin\omega t$$

In general $\quad E_i = E_i e^{-j\beta x} = E_i e^{j(\omega t-\beta x)}$

$$E_r = E_r e^{+j(\beta x+\delta)} = E_r e^{j(\omega t+\beta x+\delta)}$$

$$E_T(x) = E_i + E_r$$

$$= E_i e^{j(\omega t-\beta x)} + E_r e^{j(\omega t+\beta x+\pi)}$$

The amplitudes of E_i & E_r are equal

$$E_T(x) = E_i\left[e^{j(\omega t-\beta x)} + e^{j(\omega t+\beta x+\pi)}\right]$$

Taking the real part

$$E_T(x) = E_i\left[\cos(\omega t-\beta x) + \cos(\omega t+\beta x+\pi)\right]$$

$$= E_i\left[\cos(\omega t-\beta x) - \cos(\omega t+\beta x)\right]$$

$$= 2E_i \sin\beta x \sin\omega t$$

Therefore $\quad \tilde{E}_T(x,t) = 2E_i \sin\beta x \sin\omega t$

→ This is the equation of a standing wave in terms of electric field. stands → does not travel or progress or propagate.

*** The electric field strength be reversed in phase on reflection in order to produce zero resultant field at the surface, it follows that the magnetic field strength must be reflected without reversal of phase. If both magnetic and electric field strengths were reversed, there would be no reversal (change) of direction of energy propagation. $\therefore \boxed{H_i = H_r}$

→ The totally reflected wave combines with the incident wave to form a standing wave.

→ The magnitude of the electric field varies sinusoidally with distance (x) from the reflecting plane.

→ It is zero at the surface ($\sigma = \infty \rightarrow E = 0$) and at multiples of half wavelength from the surface

$$\vec{E}_T(x,t) = 2E_i \sin\beta x \sin\omega t$$

$$= 2E_i \sin\left(\frac{2\pi}{\lambda} \times \frac{\lambda}{2}\right) \sin\omega t \qquad \boxed{\beta = \frac{2\pi}{\lambda}}$$

$$= 2E_i \sin\pi \sin\omega t = 0$$

→ It has a maximum value of twice the electric field strength of the incident wave at distances from the surface that are odd multiples of a quarter wave length.

$$\vec{E}_T(x,t) = 2E_i \sin\left(\frac{2\pi}{\lambda} \times \frac{\lambda}{4}\right) \sin\omega t = \boxed{2E_i}$$

III'ly for the magnetic field

$$H_T(x) = H_i e^{-j\beta x} + H_r e^{j\beta x}$$

$$= H_i \left[e^{-j\beta x} + e^{j\beta x} \right] \quad [H_i = H_r]$$

$$= 2H_i \cos\beta x$$

$$\tilde{H}_T(x) = Re\left\{ 2H_i \cos\beta x \, e^{j\omega t} \right\}$$

$$= \boxed{2H_i \cos\beta x \cos\omega t}$$

→ This is the eqn of a standing wave in terms of magnetic field

→ It has a maximum value of $2H_i$ at the surface and at distances which are odd multiples of $\lambda/2$ from the surface. where as zero points occur at odd multiples of a quarter wave length from the surface.

Reflection by a perfect dielectric (insulator) - Normal incidence

→ when a plane em wave is incident normally on the surface of a perfect dielectric, part of the energy is transmitted and part of it is reflected.

→ A perfect dielectric is one with zero conductivity so that there is no loss (or) absorption of power in propagation through the dielectric.

$$\alpha = \left[\omega \frac{\mu\epsilon}{2}\left[\sqrt{1+\frac{\sigma^2}{\omega^2\epsilon^2}}-1\right]\right]^{1/2}$$

$\alpha = 0$ → There is no attenuation (or) there is no loss

$\sigma = 0 \Rightarrow \alpha = 0$

→ consider a plane wave travelling in the x-direction incident on a boundary that is parallel to the $x=0$ plane.

Let E_i be the electric field strength of the incident wave striking the boundary and

→ E_r be the electric field strength of the reflected wave leaving the boundary in the 1st medium

→ E_t be the electric field strength of the transmitted wave propagated in to the 2nd medium.

→ Similar subscripts will be applied to the magnetic field strength 'H'.

Let ϵ_1 and μ_1 be the constants of the 1st medium and ϵ_2 and μ_2 be the constants of the 2nd medium

W.K.T $\eta = \dfrac{E}{H}$ 1 represents medium ①, 2 " " ②

$$\eta_1 = \frac{E_i}{H_i} \quad \text{— ①} \qquad \Rightarrow \frac{-E_r}{H_r} = \eta_1 \quad \text{— ②} \qquad \eta_2 = \frac{E_t}{H_t} \quad \text{— ③}$$

and $H_i + H_r = H_t$ — ④

$E_i + E_r = E_t$ — ⑤

from eqn ④

$$H_i + H_r = \frac{1}{\eta_2}(E_t) = \frac{1}{\eta_2}(E_i + E_r)$$

$$H_i = \frac{E_i}{\eta_1} \quad \& \quad H_r = \frac{-E_r}{\eta_1}$$

Substitute in the above eqn

$$\frac{1}{\eta_1}(E_i - E_r) = \frac{1}{\eta_2}(E_i + E_r) \Rightarrow (E_i - E_r)\eta_2 = \eta_1(E_i + E_r)$$

$$E_i(\eta_2 - \eta_1) = (\eta_2 + \eta_1)E_r$$

$$\boxed{\frac{E_r}{E_i} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}} \rightarrow \text{Reflection coefficient}$$

Also $\frac{E_t}{E_i} = \frac{E_r + E_i}{E_i} = 1 + \frac{E_r}{E_i} = 1 + \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$

Transmission Coefficient $\leftarrow \boxed{\frac{E_t}{E_i} = \frac{2\eta_2}{\eta_2 + \eta_1}}$

lll'y * $\frac{H_r}{H_i} = \frac{-E_r}{E_i} = \frac{\eta_1 - \eta_2}{\eta_1 + \eta_2}$

$$\frac{H_t}{H_i} = \frac{H_i + H_r}{H_i} = 1 + \frac{H_r}{H_i} = \boxed{\frac{2\eta_1}{\eta_1 + \eta_2}}$$

→ * $\frac{E_r}{E_i} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{\sqrt{\frac{\mu_2}{\varepsilon_2}} - \sqrt{\frac{\mu_1}{\varepsilon_1}}}{\sqrt{\frac{\mu_2}{\varepsilon_2}} + \sqrt{\frac{\mu_1}{\varepsilon_1}}}$ $\qquad \eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}}$, $\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}$

Practically the permeability values of different dielectric media are equal to freespace i.e $\mu_1 = \mu_2 = \mu_0$

$\therefore$ * $\frac{E_r}{E_i} = \frac{\sqrt{\varepsilon_1} - \sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}}$ $\qquad$ * $\frac{H_r}{H_i} = \frac{\sqrt{\varepsilon_2} - \sqrt{\varepsilon_1}}{\sqrt{\varepsilon_2} + \sqrt{\varepsilon_1}}$

* $\frac{E_t}{E_i} = \frac{2\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}}$ $\qquad$ * $\frac{H_t}{H_i} = \frac{2\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}}$

Reflection by a perfect insulator - oblique incidence.

Angled (or) Slantly

Whenever a wave is incident obliquely on the interface between two media, it is necessary to consider seperately two special cases.

1. Horizontal polarization
2. Vertical polarization

Backside

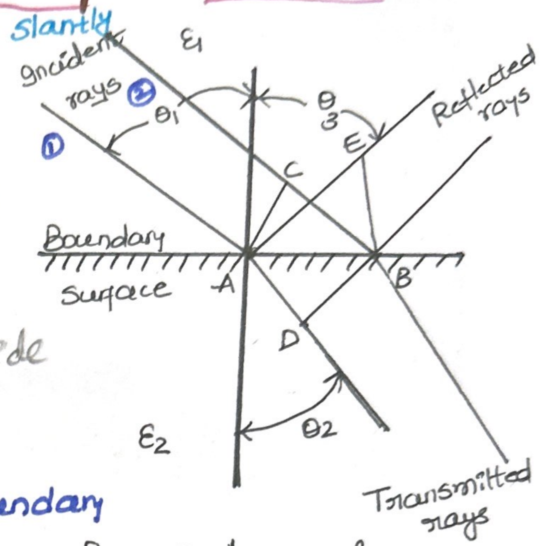


fig: Reflection & Refraction

→ If a plane wave is incident upon a boundary surface that is not parallel to the plane containing E and H, the boundary conditions are more complex.

→ Again part of the wave will be transmitted and part of it is reflected. but in this case the transmitted wave will be refracted with an angle θ_2 with the normal (i.e the direction of propagation will be altered). consider fig:

→ The incident ray ② travels the distance CB in medium ① where as transmitted ray ① travels from (the distance) AD in medium ②.

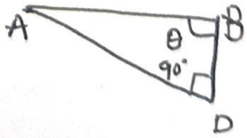
→ If v_1 is the velocity of the wave in medium ① and
v_2 " " " " " " " ②

from fig next page

Assume $\theta_1 = \theta_3$

$$\therefore \frac{v_1}{v_2} = \frac{CB}{AD}$$

$\theta_1 \rightarrow$ angle of incidence
$\theta_2 \rightarrow$ " " refraction
$\theta_3 \rightarrow$ " " reflection



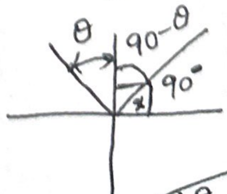
$\sin\theta_2 = \frac{AD}{AB}$

from Δ^{le} ABC

$$\frac{CB}{AB} = \sin\theta_1$$

from Δ^{le} ABD

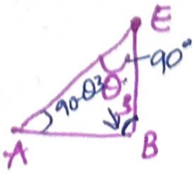
$$\frac{AD}{AB} = \sin\theta_2$$

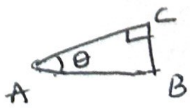


$90 = 90 - \theta + x$
$x = \theta$

$CB = AB\sin\theta_1$

$\sin\theta_3 = \frac{AE}{AB}$





$$\frac{v_1}{v_2} = \frac{AB\sin\theta_1}{AB\sin\theta_2} = \frac{\sin\theta_1}{\sin\theta_2}$$

The distance travelled by incident wave is = " " " " reflected wave

i.e $CB = AE$

In terms of the constants of the media, v_1 & v_2 are given by

$AB\sin\theta_1 = AB\sin\theta_3$

$\boxed{\theta_1 = \theta_3}$

$$v_1 = \frac{1}{\sqrt{\mu_1\epsilon_1}} \quad \text{and} \quad v_2 = \frac{1}{\sqrt{\mu_2\epsilon_2}}$$

$\mu_1 = \mu_2 = \mu_0$

$$\therefore \quad \frac{\sin\theta_1}{\sin\theta_2} = \sqrt{\frac{\epsilon_2}{\epsilon_1}} \rightarrow (1)$$

The angle of incidence is equal to the angle of reflection; the angle of incidence is related to the angle of refraction by eqn (1) which in optics is known as Law of sines (or) **Snell's Law.**

The product of $E \cdot H = \frac{V}{m} \cdot \frac{A}{m} = P/m^2 =$ Power / area

$$E \cdot H = E \cdot \frac{E}{\eta} = \frac{E^2}{\eta} \quad (\eta = \frac{E}{H})$$

$|E||\frac{E}{\eta}|\cos\theta_1 \Leftarrow |E||H|\cos\theta$

The power in the incident wave $= \frac{E_i^2}{\eta_1}$ (medium 1)

The component of this along the normal $= \frac{E_i^2}{\eta_1}\cos\theta_1$

The power in the reflected wave $= \frac{E_r^2}{\eta_1}$

The component of this along the normal $= \frac{E_r^2}{\eta_1}\cos\theta_1$ $(\theta_1 = \theta_3)$

The power in the transmitted wave $= \frac{E_t^2}{\eta_2}$

The component of this along the normal $= \frac{E_t^2}{\eta_2}\cos\theta_2$

According to law of conservation of energy

$$\frac{E_i^2}{\eta_1}\cos\theta_1 = \frac{E_r^2}{\eta_1}\cos\theta_1 + \frac{E_t^2}{\eta_2}\cos\theta_2$$

[incident wave energy is the sum of the reflected and transmitted wave energy]

$$\frac{E_i^2}{\eta_1}\cos\theta_1 = \frac{E_r^2}{\eta_1}\cos\theta_1 + \frac{E_t^2}{\eta_2}\cos\theta_2$$

Dividing with E_i^2

$$\frac{1}{\eta_1}\cos\theta_1 = \frac{E_r^2}{E_i^2\eta_1}\cos\theta_1 + \frac{E_t^2}{\eta_2 E_i^2}\cos\theta_2$$

$$\frac{E_r^2}{E_i^2\eta_1}\cos\theta_1 = \frac{1}{\eta_1}\cos\theta_1 - \frac{E_t^2}{\eta_2 E_i^2}\cos\theta_2$$

$$\frac{E_r^2}{E_i^2} = 1 - \frac{\eta_1}{\eta_2}\frac{E_t^2}{E_i^2}\frac{\cos\theta_2}{\cos\theta_1}$$

$\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}$; $\eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}}$

If $\mu_1 = \mu_2 = \mu_0$

$$\frac{E_r^2}{E_i^2} = 1 - \frac{\sqrt{\varepsilon_2}\,E_t^2\cos\theta_2}{\sqrt{\varepsilon_1}\,E_i^2\cos\theta_1} \quad \text{--- (2)}$$

Case 1: Horizontal (or perpendicular) Polarization:-

In this case the electric vector 'E' is $\perp$ar to the plane of incidence and parallel to the boundary surface.

According to the boundary conditions $E_i + E_r = E_t$

$$\frac{E_t}{E_i} = 1 + \frac{E_r}{E_i} \quad \text{--- (3)}$$

The plane of incidence is the plane containing the incident ray and the normal to the surface.

Substitute eqn (3) in to eqn (2)

$$\frac{E_r^2}{E_i^2} = 1 - \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\left[1+\left(\frac{E_r}{E_i}\right)\right]^2\frac{\cos\theta_2}{\cos\theta_1}$$

$$1 - \frac{E_r^2}{E_i^2} = \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\left[1+\frac{E_r}{E_i}\right]^2\frac{\cos\theta_2}{\cos\theta_1}$$

$$\cancel{\left(1+\frac{E_r}{E_i}\right)}\left(1-\frac{E_r}{E_i}\right) = \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\left[1+\frac{E_r}{E_i}\right]^{\cancel{2}}\frac{\cos\theta_2}{\cos\theta_1}$$

$$1 - \frac{E_r}{E_i} = \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\left[1+\frac{E_r}{E_i}\right]\frac{\cos\theta_2}{\cos\theta_1}$$

$$\left(1 - \frac{E_r}{E_i}\right) = \left[\frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}} + \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\frac{E_r}{E_i}\right]\frac{\cos\theta_2}{\cos\theta_1}$$

$$1 - \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\frac{\cos\theta_2}{\cos\theta_1} = \frac{E_r}{E_i}\left[1 + \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\frac{\cos\theta_2}{\cos\theta_1}\right]$$

$$\Rightarrow \frac{E_r}{E_i} = \frac{1 - \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\frac{\cos\theta_2}{\cos\theta_1}}{1 + \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\frac{\cos\theta_2}{\cos\theta_1}} = \frac{\sqrt{\varepsilon_1}\cos\theta_1 - \sqrt{\varepsilon_2}\cos\theta_2}{\sqrt{\varepsilon_1}\cos\theta_1 + \sqrt{\varepsilon_2}\cos\theta_2}$$

from snell's Law

$$\frac{\sin\theta_1}{\sin\theta_2} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \Rightarrow \sin\theta_2 = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\sin\theta_1$$

$$\cos\theta_2 = \sqrt{1 - \frac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_1}$$

$$\Rightarrow \frac{E_r}{E_i} = \frac{\sqrt{\varepsilon_1}\cos\theta_1 - \sqrt{\varepsilon_2}\sqrt{1 - \frac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_1}}{\sqrt{\varepsilon_1}\cos\theta_1 + \sqrt{\varepsilon_2}\sqrt{1 - \frac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_1}}$$

$$= \frac{\sqrt{\varepsilon_1}\cos\theta_1 - \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_2}}\sqrt{\varepsilon_2 - \varepsilon_1\sin^2\theta_1}}{\sqrt{\varepsilon_1}\cos\theta_1 + \sqrt{\varepsilon_2 - \varepsilon_1\sin^2\theta_1}}$$

$$= \frac{\cos\theta_1 - \sqrt{\frac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_1}}{\cos\theta_1 + \sqrt{\frac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_1}}$$

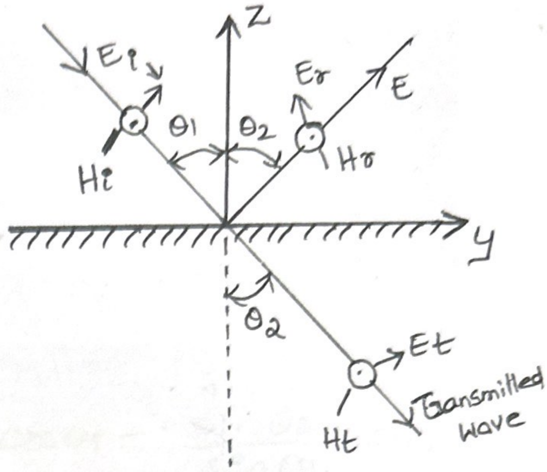
The above eqn gives the ratio of reflected to incident electric field strength for the case of a $\perp$ar polarised wave. and this eqn is the reflection coefficient for horizontal polarization.

Case 2: Vertical Polarization:

The magnetic vector is parallel to the boundary surface and the electric vector is parallel to the plane of inc...

The tangential component of E_i is

$$E_i \sin(90-\theta_1) = E_i \cos\theta_1$$



The tangential component of E_r is

$$-E_r \sin(90-\theta_1) = -E_r \cos\theta_1$$

The tangential component of E_t is

$$E_t \sin(90-\theta_2) = E_t \cos\theta_2$$

$\Rightarrow$ from the boundary condition $E_t = E_i + E_r$

$$E_t \cos\theta_2 = E_i \cos\theta_1 - E_r \cos\theta_1$$

$$\frac{E_t}{E_i} = \frac{\cos\theta_1}{\cos\theta_2} - \frac{E_r}{E_i}\frac{\cos\theta_1}{\cos\theta_2}$$

$$= \left[1 - \frac{E_r}{E_i}\right]\frac{\cos\theta_1}{\cos\theta_2}$$

Substitute eqn (the above) in eqn (2)

$$\frac{E_r^2}{E_i^2} = 1 - \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\frac{\cos^2\theta_1}{\cos^2\theta_2}\left(1 - \frac{E_r}{E_i}\right)^2 \frac{\cos\theta_2}{\cos\theta_1}$$

$$= 1 - \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\frac{\cos\theta_1}{\cos\theta_2}\left(1 - \frac{E_r}{E_i}\right)^2$$

$$\cancel{\left(1 - \frac{E_r}{E_i}\right)}\left(1 + \frac{E_r}{E_i}\right) = \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\frac{\cos\theta_1}{\cos\theta_2}\left(1 - \frac{E_r}{E_i}\right)^{\cancel{2}}$$

$$\frac{E_r}{E_i}\left[1 + \frac{\sqrt{\varepsilon_2}\cos\theta_1}{\sqrt{\varepsilon_1}\cos\theta_2}\right] = \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\frac{\cos\theta_1}{\cos\theta_2} - 1$$

$$\frac{E_r}{E_i}\left[\frac{\sqrt{\varepsilon_1}\cos\theta_2 + \sqrt{\varepsilon_2}\cos\theta_1}{\sqrt{\varepsilon_1}\cos\theta_1}\right] = \frac{\sqrt{\varepsilon_2}\cos\theta_1 - \sqrt{\varepsilon_1}\cos\theta_2}{\sqrt{\varepsilon_1}\cos\theta_2}$$

$$\Rightarrow \frac{E_r}{E_i} = \frac{\sqrt{\varepsilon_2}\cos\theta_1 - \sqrt{\varepsilon_1}\cos\theta_2}{\sqrt{\varepsilon_1}\cos\theta_2 + \sqrt{\varepsilon_2}\cos\theta_1}$$

from Snell's Law

$$\cos\theta_2 = \sqrt{1 - \frac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_1}$$

$$\frac{E_r}{E_i} = \frac{\sqrt{\epsilon_2}\cos\theta_1 - \sqrt{\epsilon_1}\sqrt{1-\frac{\epsilon_1}{\epsilon_2}\sin^2\theta_1}}{\sqrt{\epsilon_2}\cos\theta_1 + \sqrt{\epsilon_1}\sqrt{1-\frac{\epsilon_1}{\epsilon_2}\sin^2\theta_1}}$$

divide by $\sqrt{\epsilon_1}$

$$= \frac{\sqrt{\frac{\epsilon_2}{\epsilon_1}}\cos\theta_1 - \sqrt{1-\frac{\epsilon_1}{\epsilon_2}\sin^2\theta_1}}{\sqrt{\frac{\epsilon_2}{\epsilon_1}}\cos\theta_1 + \sqrt{1-\frac{\epsilon_1}{\epsilon_2}\sin^2\theta_1}}$$

$$= \frac{\sqrt{\frac{\epsilon_2}{\epsilon_1}}\cos\theta_1 - \sqrt{\frac{\epsilon_1}{\epsilon_2}}\sqrt{\frac{\epsilon_2}{\epsilon_1}-\sin^2\theta_1}}{\sqrt{\frac{\epsilon_2}{\epsilon_1}}\cos\theta_1 + \sqrt{\frac{\epsilon_1}{\epsilon_2}}\sqrt{\frac{\epsilon_2}{\epsilon_1}-\sin^2\theta_1}}$$

$$= \frac{\frac{\epsilon_2}{\epsilon_1}\cos\theta_1 - \sqrt{\frac{\epsilon_2}{\epsilon_1}-\sin^2\theta_1}}{\frac{\epsilon_2}{\epsilon_1}\cos\theta_1 + \sqrt{\frac{\epsilon_2}{\epsilon_1}-\sin^2\theta_1}}$$

The above eqn gives the reflection coefficient for parallel (or) vertical polarization i.e the ratio of reflected to incident electric field strength when E is parallel to the plane of incidence.

Proove for parallel polarization that $\frac{E_r}{E_i} = \frac{\tan(\theta_1-\theta_2)}{\tan(\theta_1+\theta_2)}$ &

Proof (see inside)

for ⊥ar polarization that $\frac{E_r}{E_i} = \frac{\sin(\theta_2-\theta_1)}{\sin(\theta_2+\theta_1)}$

⊥ar polarization:

Consider $\frac{E_r}{E_i} = \frac{\sqrt{\epsilon_1}\cos\theta_1 - \sqrt{\epsilon_2}\cos\theta_2}{\sqrt{\epsilon_1}\cos\theta_1 + \sqrt{\epsilon_2}\cos\theta_2} = \frac{\cos\theta_1 - \sqrt{\frac{\epsilon_2}{\epsilon_1}}\cos\theta_2}{\cos\theta_1 + \sqrt{\frac{\epsilon_2}{\epsilon_1}}\cos\theta_2}$

from snell's law

$\frac{\sin\theta_1}{\sin\theta_2} = \sqrt{\frac{\epsilon_2}{\epsilon_1}} \Rightarrow \frac{E_r}{E_i} = \frac{\cos\theta_1 - \frac{\sin\theta_1}{\sin\theta_2}\cos\theta_2}{\cos\theta_1 + \frac{\sin\theta_1}{\sin\theta_2}\cos\theta_2}$

$$\Rightarrow \frac{E_r}{E_i} = \frac{\sin\theta_2\cos\theta_1 - \sin\theta_1\cos\theta_2}{\sin\theta_2\cos\theta_1 + \sin\theta_1\cos\theta_2} = \frac{\sin(\theta_2-\theta_1)}{\sin(\theta_2+\theta_1)}$$

Brewster angle :-

If any wave incident at the boundary surface with some an angle and if it is not producing the reflected wave then that angle is called Brewster angle. Brewster angle exists only for vertical polarization.

It is the angle of incidence for which there is no reflections. Complete transmission takes place.

At the Brewster angle $E_r = 0$

For vertical polarization

$$\frac{E_r}{E_i} = \frac{\frac{\varepsilon_2}{\varepsilon_1}\cos\theta_1 - \sqrt{\frac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_1}}{\frac{\varepsilon_2}{\varepsilon_1}\cos\theta_1 + \sqrt{\frac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_1}} = 0$$

$$\left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2 \cos^2\theta_1 = \frac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_1$$

$$\left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2 [1 - \sin^2\theta_1] = \frac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_1$$

$$\sin^2\theta_1 \left(1 - \frac{\varepsilon_2^2}{\varepsilon_1^2}\right) = \frac{\varepsilon_2}{\varepsilon_1} - \frac{\varepsilon_2^2}{\varepsilon_1^2}$$

$$\sin^2\theta_1 = \frac{\varepsilon_2\varepsilon_1^2 - \varepsilon_1\varepsilon_2^2}{\varepsilon_1 \cancel{\varepsilon_1^2}} \times \frac{\cancel{\varepsilon_1^2}}{\varepsilon_1^2 - \varepsilon_2^2}$$

$$= \frac{\varepsilon_2\varepsilon_1 - \varepsilon_2^2}{\varepsilon_1^2 - \varepsilon_2^2} = \frac{\varepsilon_2(\cancel{\varepsilon_1 - \varepsilon_2})}{(\varepsilon_1 + \varepsilon_2)(\cancel{\varepsilon_1 - \varepsilon_2})}$$

$$\sin^2\theta_1 = \frac{\varepsilon_2}{\varepsilon_1 + \varepsilon_2}$$

$$\sin^2\theta_1 = \frac{\varepsilon_2}{\varepsilon_1 + \varepsilon_2}$$

Ill^{ly} $\cos^2\theta_1 = \frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2}$

$$\tan\theta_1 = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

$\theta_1 = \tan^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$ $\rightarrow$ At this angle, which is called the Brewster angle, there is not reflected wave when the incident wave is parallel (or vertically) polarized

For horizontal polarization $\frac{E_r}{E_i} = \frac{\cos\theta_1 - \sqrt{\frac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_1}}{\cos\theta_1 + \sqrt{\frac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_1}}$

In this case it is not possible to find the value of 'θ'.

Poynting theorem (or) Poynting Vector :-

→ In order to find the power flow associated with an electromagnetic wave, it is necessary to develop a power theorem for the electromagnetic field known as the poynting theorem.

Maxwell's first eqn : $\nabla \times H = J + \dot{D}$

$= J + \varepsilon \dot{E}$

$\Rightarrow J = (\nabla \times H) - \varepsilon \dot{E}$

2nd eqn

$\nabla \times E = -\dot{B}$

$= -\mu \dot{H}$ — ①

multiply throught by E

$$E \cdot J = E \cdot (\nabla \times H) - E \cdot \varepsilon \dot{E}$$

The vector identity $\nabla \cdot (E \times H) = H \cdot (\nabla \times E) - E \cdot (\nabla \times H)$

$$\Rightarrow E \cdot (\nabla \times H) = H \cdot (\nabla \times E) - \nabla \cdot (E \times H)$$

$$\therefore E \cdot J = H \cdot (\nabla \times E) - \nabla \cdot (E \times H) - E \cdot (\varepsilon \dot{E})$$

$$= -H \cdot \mu \dot{H} - \nabla \cdot (E \times H) - E \cdot (\varepsilon \dot{E})$$ from eqn ①

We know that $\frac{\partial}{\partial t}(H^2) = 2H \frac{\partial H}{\partial t} = 2H\dot{H}$

$$\Rightarrow H \cdot \dot{H} = \frac{1}{2} \frac{\partial}{\partial t}(H^2)$$

$$E \cdot J = -\frac{\mu}{2} \frac{\partial}{\partial t}(H^2) - \nabla \cdot (E \times H) - E \cdot (\varepsilon \dot{E})$$

$$= \frac{\mu}{2} \frac{\partial}{\partial t}(H^2) - \nabla \cdot (E \times H) - \frac{\varepsilon}{2} \frac{\partial}{\partial t}(E^2)$$

Integrating over a volume

$$\int_v E \cdot J \, dv = -\int_v \frac{\mu}{2} \frac{\partial}{\partial t}(H^2) dv - \int_v \nabla \cdot (E \times H) dv - \int_v \frac{\varepsilon}{2} \frac{\partial E^2}{\partial t} dv$$

$$\Rightarrow \int_v E \cdot J \, dv = -\frac{\partial}{\partial t} \int_v \left(\frac{\mu}{2} H^2 + \frac{\varepsilon}{2} E^2 \right) dv - \int_v \nabla \cdot (E \times H) dv$$

By divergence theorem

$$\int_v \nabla \cdot (E \times H) dv = \oint_s (E \times H) ds$$

Poynting Vector :-

The Poynting vector is defined as the cross product of the electric field vector $\vec{E}$ and the magnetic field vector of an electromagnetic wave. Thus

$$\vec{P} = \vec{E} \times \vec{H} \quad \text{Watts}/m^2$$

Where $\vec{P}$ = Poynting vector in watts/m^2

$\vec{E}$ = electric field vector in V/m

$\vec{H}$ = Magnetic field vector in A/m

The Poynting vector gives instantaneous power density of the wave. The direction of $\downarrow$ is $\perp$ar to the plane of the field and follows the right hand screw law.

The first term $\int_v (\vec{E} \cdot \vec{J})\, dv$ = Total power dissipated in a volume when the electric field vector produces a current density J in the conducting medium. It represents the power dissipated due to ohmic (I^2R) loss of the conductor.

$\vec{E} \cdot \vec{J}$ represents power density in W/m^3.

The Second term $\frac{d}{dt} \int_v \left[\frac{\mu H^2}{2} + \frac{\epsilon E^2}{2}\right] dv$ .. Electric field

$\hookrightarrow$ Energy density stored in the magnetic field

$\hookrightarrow$ rate at which the stored energy in the volume is decreasing.

The third term $-\oint_S (\vec{E} \times \vec{H})\, \vec{ds}$ = The rate of flow of energy inward through the surface of the volume.

$$E \cdot J = \frac{E \cdot I}{A} = W/m^3$$

(J: A/m^2, E: V/m)

The vector product $E \times H$ at any point is a measure of the rate of energy flow / unit area at that point.

The current flows along the surface of the conductor due to tangential component of the electric field vector.

$\bar{J_s}$ can be obtained by integrating the current density $\bar{J}$ from 0 to ∞

$$\int_v E\cdot J\,dv = -\frac{d}{dt}\int_v \left(\frac{\mu}{2}H^2 + \frac{\varepsilon}{2}E^2\right)dv - \oint_S (E\times H)\,ds$$

$\int_v E\cdot J\,dv$ = total instantaneous power dissipated in the volume.

$-\frac{d}{dt}\int_v \left(\frac{1}{2}\mu H^2 + \frac{1}{2}\varepsilon E^2\right)dv$ = total Energy stored in the volume.

$\frac{1}{2}\mu H^2 = w_m$ = Energy stored in the magnetic field

$\frac{1}{2}\varepsilon E^2 = w_E$ = " " " " Electric field

→ -Ve time derivative represents the rate at which the stored Energy in the volume is decreasing.

From Law of Conservation Of Energy

The rate of Energy dissipation with in the volume 'V' must equal the rate at which the stored energy in volume is decreasing + the rate at which energy is entering the Volume 'V' from outside.

from the definition

$-\oint_S (E\times H)\,ds$ = Rate of flow of energy inwards

(+ve) $\oint_S (E\times H)\,ds$ = " " " " " outwards through the surface enclosing the volume. so the total power flowing out of the volume is $\int_S (E\times \vec{H})\,ds$ watts

The cross product $(E\times H)$ is known as the poynting vector 'P'

⇒ $P = E\times H$ has the units of watts/m^2

Poynting theorem statement: The vector product $P = E\times H$ at any point is a measure of the rate of energy flow per unit area at that point. The direction of flow is ⊥ar to $E\times H$ in the direction of the vector $P(E\times H)$

Surface impedance :-

At high frequencies the current is confined almost entirely to a very thin sheet at the surface of the conductor. This is called skin effect. The impedance of the thin sheet is called surface impedance (Z_s) and is defined by

$$\boxed{Z_s = \frac{E_{tan}}{J_s}}$$

exponentially decaying

Where E_{tan} is the electric field strength parallel to and at the surface of the conductor.

J_s is the linear current density that flows as a result of this E_{tan}.

J_s represents the total conduction current/meter width flowing in the thin sheet.

*** contd → slide.

Power loss in a plane conductor :-

The complex poynting vector is defined as

$$P = \frac{1}{2}(E \times H^*)$$

$$P_{avg} = \frac{1}{2} Re\{E \times H^*\}$$

avg part of the power flow per square meter

$$P_{react} = \frac{1}{2} Im\{E \times H^*\}$$

Averge (or) real power flow per unit area normal to the surface is

$$P_n(real) = \frac{1}{2} Re\{E_{tan} \times H_{tan}^*\}$$

For good conductors $\eta = \sqrt{\frac{\omega\mu}{\sigma}} \angle 45°$

Inside the conductor E_{tan} is related to H_{tan} by $\eta_m = \frac{E_{tan}}{H_{tan}}$

→ m is used to indicate the quantities inside the metallic conductor.

$$\eta = \sqrt{\frac{\omega\mu}{\sigma}} \angle 45^\circ \qquad \eta = \frac{E}{H}$$

$$\frac{E_{tan}}{H_{tan}} = \sqrt{\frac{\omega\mu}{\sigma}} \angle 45^\circ \Rightarrow E_{tan} = H_{tan}\sqrt{\frac{\omega\mu}{\sigma}} \angle 45^\circ$$

E_{tan} leads H_{tan} by 45° in time phase

$$\rightarrow P_n = \frac{1}{2}|E_{tan}||H_{tan}|\sin 45^\circ \qquad E\times H = |E||H| \sin 45^\circ$$

$$= \frac{1}{2\sqrt{2}}|E_{tan}||H_{tan}| = \frac{1}{2\sqrt{2}}|H_{tan}|^2|\eta_m|$$

For a conductor which has a thickness very much greater than the skindepth δ, $Z_s = \eta_m$ of the conductor

$\sigma = \infty$ conductor

$J = \sigma E$

$$\rightarrow P_n = \frac{1}{2\sqrt{2}}|H_{tan}|^2|Z_s| \text{ watts}/m^2$$

From the magnetic boundary conditions

$E_{t2} = 0 \Rightarrow H_{2t} = 0$

$\eta = \frac{E}{H}$

$$K = H_{t1} - H_{t2} \Rightarrow K = H_{t1} = H_{tan}$$

$\boxed{\text{If } K = J_s}$ $H_{t1} = H_{tan} = J_s$

Surface current density.

$$\rightarrow P_n = \frac{1}{2\sqrt{2}}|J_s|^2|Z_s| \text{ watts}/m^2$$

In the above expressions, E_{tan}, H_{tan} and J_s are peak values. In terms of effective values

$$E_{eff} = \frac{E_{tan}}{\sqrt{2}}$$

$$H_{eff} = \frac{H_{tan}}{\sqrt{2}}$$

$$J_{eff} = \frac{J_s}{\sqrt{2}}$$

Total internal reflection & critical angle – See the notes backside

Surface impedance :-

→ defined as the ratio of the tangential component of the electric field to the surface current density at the surface. It can be expressed as

$$Z_S = \frac{E_{tan}}{J_S} = \frac{E_{tan}}{K} = \frac{\gamma}{\sigma}\ \Omega$$

Where E_{tan} = tangential component of the electric field vector V/m, → E which is ∥ to and at surface of the conductor

$K = J_S$ = total surface current density A/m

γ = Propagation constant & σ = conductivity of the sheet.

Proof :-

Consider an em wave travelling along a conducting plate of thickness 't' lying at z=0 plane as shown in fig:

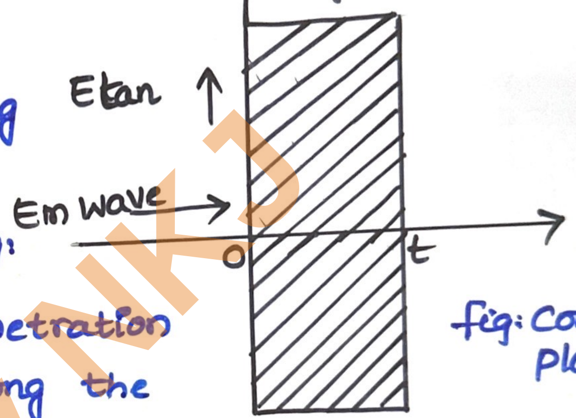


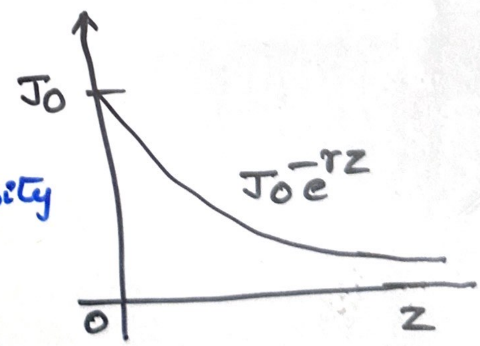
fig: Conducting plate

Assume that the depth of penetration of the current in to the plate along the z-direction is less than the plate thickness 't'.

Let 'J' be the current density. It can be expressed as $J = J_0 e^{-\gamma z}$

Where J_0 = current density in A/m² at the surface at z=0 and γ = propagation Constant.

The variation of current density inside the plate is shown in fig:



If the current is distributed along the conductor surface, the surface current density

$$J_S = \int_0^\infty J\,dz = \int_0^\infty J_0 e^{-\gamma z}\,dz$$

$$J_S = J_0\left[\frac{e^{-\gamma z}}{-\gamma}\right]_0^\infty \quad (\text{or}) \quad J_S = \frac{J_0}{\gamma}\ \text{A/m}$$

Poynting theorem Proof:

consider Maxwell's Eqn $\nabla \times H = J + \dot{D}$

$$= J + \varepsilon\dot{E}$$

$$J = (\nabla \times H) - \varepsilon\dot{E}$$

$$\nabla \times E = -\dot{B}$$
$$= -\mu\dot{H}$$

multiply through out by E

$$E \cdot J = E \cdot (\nabla \times H) - E \cdot \varepsilon\dot{E}$$

The Vector identity $\nabla \cdot (E \times H) = H \cdot (\nabla \times E) - E \cdot (\nabla \times H)$

$$E \cdot (\nabla \times H) = H \cdot (\nabla \times E) - \nabla \cdot (E \times H)$$

$$E \cdot J = H \cdot (\nabla \times E) - \nabla \cdot (E \times H) - E \cdot \varepsilon\dot{E}$$

$$= -H \cdot \mu\dot{H} - \nabla \cdot (E \times H) - E \cdot \varepsilon\dot{E}$$

W.K.T $\frac{d}{dt}(H^2) = 2H\frac{dH}{dt} = 2H\dot{H}$

$$\rightarrow H\dot{H} = \frac{1}{2}\frac{d}{dt}(H^2)$$

$$\therefore \; E \cdot J = -\frac{\mu}{2}\frac{d}{dt}(H^2) - \nabla \cdot (E \times H) - \frac{\varepsilon}{2}\frac{d}{dt}(E^2)$$

Integrating over a volume

$$\int_v (E \cdot J)\,dv = -\int_v \frac{\mu}{2}\frac{d}{dt}(H^2)\,dv - \int_v \nabla \cdot (E \times H)\,dv - \int_v \frac{\varepsilon}{2}\frac{d}{dt}(E^2)\,dv$$

$$\Rightarrow \int_v E \cdot J\,dv = -\frac{d}{dt}\int_v \left(\frac{\mu}{2}H^2 + \frac{\varepsilon}{2}E^2\right)dv - \int_v \nabla \cdot (E \times H)\,dv$$

By divergence theorem

$$\int_v \nabla \cdot (E \times H)\,dv = \oint_s (E \times H)\,ds$$

$$\therefore \int_v E \cdot J\,dv = -\frac{d}{dt}\int_v \left[\frac{\mu}{2}H^2 + \frac{\varepsilon}{2}E^2\right]dv - \oint_s (E \times H)\,ds$$

The current density at the surface is

$J = \sigma E$

$J_0 = \sigma E_{tan}$

$$Z_s = E_{tan} / J_s = \frac{J_0}{\sigma} \Big/ \frac{J_0}{\gamma}$$

$$\boxed{Z_s = \frac{\gamma}{\sigma}\ \Omega}$$

W.K.T $\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$

For a good conductor $\sigma >> \omega\varepsilon$

$$\gamma = \sqrt{j\omega\mu\sigma}$$

Surface impedance $Z_s = \frac{\sqrt{j\omega\mu\sigma}}{\sigma} = \sqrt{\frac{\omega\mu}{\sigma}}\ \angle 45^\circ$

W.K.T the characteristic impedance $\eta = \sqrt{\frac{\omega\mu}{\sigma}}\ \angle 45^\circ$

Therefore for a good conductor, the surface impedance is equal to the characteristic impedance. $\boxed{Z_s = \eta}$

Power Loss in a plane conductor:

The complex poynting vector is defined as $P_{com} = \frac{1}{2} E \times H^*$

$= P_{avg} + P_{reac}$

$P_{avg} = \frac{1}{2} \mathrm{Re}\{E \times H^*\}$; $P_{reac} = \frac{1}{2} \mathrm{Im}\{E \times H^*\}$

$P_{normal} = \frac{1}{2} \mathrm{Re}\{E_{tan} \times H_{tan}^*\}$ but $\eta = \frac{E}{H} = \frac{E_{tan}}{H_{tan}} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$

For good conductors $\sigma >> \omega\varepsilon$ $\quad E_{tan} = H_{tan}\sqrt{\frac{\omega\mu}{\sigma}}\ \angle 45^\circ$

$\therefore$ E_{tan} Leads H_{tan} by 45°

$$P_n = \frac{1}{2}|E_{tan}||H_{tan}| \sin 45^\circ = \frac{1}{2\sqrt{2}} \eta (H_{tan}^2) = \frac{1}{2\sqrt{2}} Z_s H_{tan}^2$$

but $H_{t_1} - H_{t_2} = J_s$ [For a conductor $H_{t_2} = 0$, $H_{t_1} = J_s$]

$$P_n = \frac{1}{2\sqrt{2}} Z_s J_s^2 \quad W/m^2$$

$J = \sigma E$

$E \propto \frac{1}{\sigma}$

$\eta H \propto \frac{1}{\sigma}$

$H \propto \frac{1}{\eta\sigma}$

$\sigma \to \infty$

$H = 0$

Reflection by a Perfect dielectric (insulator) – Normal incidence.

Perfect dielectric $\sigma = 0 \Rightarrow$ $\alpha = 0$ $\quad \alpha = \sqrt{\frac{\omega\mu\varepsilon}{2}\left[\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}}-1\right]}$

that means there is no attenuation

or there is no Loss.

Let $E_i \rightarrow$ incident wave $\quad E_r$: reflected wave

$E_t \rightarrow$ transmitted wave

ε_1 & μ_1 be the constants of the 1st medium and

ε_2 & μ_2 " " " " " 2nd medium

W.K.T $\quad \eta = \frac{E}{H} \Rightarrow \eta_1 = \frac{E_i}{H_i} = \frac{-E_r}{H_r} = \eta_1$

$$\eta_2 = \frac{E_t}{H_t}$$

$E_i + E_r = E_t$

$H_i + H_r = H_t \Rightarrow H_i + H_r = \frac{E_t}{\eta_2} = \frac{1}{\eta_2}(E_i + E_r)$

$$\frac{E_i}{\eta_1} - \frac{E_r}{\eta_1} = \frac{1}{\eta_2}(E_i + E_r)$$

$$E_i\eta_2 - E_r\eta_2 = E_i\eta_1 + E_r\eta_1$$

$$E_i(\eta_2 - \eta_1) = E_r(\eta_2 + \eta_1)$$

Reflection coefficient $\leftarrow$ $\boxed{\frac{E_r}{E_i} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}}$

Transmission coefficient $= \frac{E_t}{E_i} = \frac{E_i + E_r}{E_i} = 1 + \frac{E_r}{E_i}$

$$\boxed{\frac{E_t}{E_i} = \frac{2\eta_2}{\eta_2 + \eta_1}}$$

III'ly $\quad \frac{H_r}{H_i} = \frac{-E_r}{E_i} = \frac{\eta_1 - \eta_2}{\eta_1 + \eta_2}$

$$\boxed{\frac{H_t}{H_i} = \frac{2\eta_1}{\eta_1 + \eta_2}}$$

$$\frac{E_r}{E_i} = \frac{\eta_2-\eta_1}{\eta_2+\eta_1} = \frac{\sqrt{\frac{\mu_2}{\varepsilon_2}} - \sqrt{\frac{\mu_1}{\varepsilon_1}}}{\sqrt{\frac{\mu_2}{\varepsilon_2}} + \sqrt{\frac{\mu_1}{\varepsilon_1}}}$$

Practically the permeability values of different dielectric media are equal to free space i.e $\mu_1 = \mu_2 = \mu_0$

$$\therefore \quad \frac{E_r}{E_i} = \frac{\sqrt{\varepsilon_1}-\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}+\sqrt{\varepsilon_2}} \qquad \frac{H_r}{H_i} = \frac{\sqrt{\varepsilon_2}-\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_2}+\sqrt{\varepsilon_1}}$$

$$\frac{E_t}{E_i} = \frac{2\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_1}+\sqrt{\varepsilon_2}} \qquad \frac{H_t}{H_i} = \frac{2\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}+\sqrt{\varepsilon_2}}$$

Snell's Law :-

$$\frac{\sin\theta_i}{\sin\theta_t} = \frac{V_1}{V_2} \qquad V_1 = \frac{1}{\sqrt{\mu_1\varepsilon_1}}$$

$$V_2 = \frac{1}{\sqrt{\mu_2\varepsilon_2}}$$

For dielectric media $\boxed{\mu_1 = \mu_2 = \mu_0}$

$$\frac{\sin\theta_i}{\sin\theta_t} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = \frac{\eta_2}{\eta_1} \qquad \eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}} = \sqrt{\frac{\mu_0}{\varepsilon_2}}$$

$$\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}} = \sqrt{\frac{\mu_0}{\varepsilon_1}}$$

Law of reflection :- States that a wave incident upon a reflective surface will be reflected at an angle equal to the incidence angle i.e $\boxed{\theta_i = \theta_r}$

Critical angle : The angle of incidence at which the refracted angle becomes $\pi/2$ is called the critical angle θ_c

i.e if $\theta_1 = \theta_c$ then $\theta_2 = \pi/2$

From Snell's Law $\sin\theta_c = \sqrt{\varepsilon_2/\varepsilon_1} \Rightarrow \theta_c = \sin^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$.

If the incidence angle $\theta_1 > \theta_c$, no transmission takes place. The wave completely reflects back. This phenomenon is called total internal reflection.

Slide-1

The product of $E \cdot H = \frac{V}{m} \cdot \frac{A}{m} = P/m^2 =$ Power/area

$$E \cdot H = E \cdot \frac{E}{\eta} = \frac{E^2}{\eta} \quad \left(\eta = \frac{E}{H}\right)$$

The Power in the incident wave $\frac{E_i^2}{\eta_1}$ (medium 1)

The component of this along the normal $= \frac{E_i^2 \cos\theta_1}{\eta_1}$

The Power in the reflected wave $= \frac{E_r^2}{\eta_1}$

The Component of this along the normal $= \frac{E_r^2 \cos\theta_1}{\eta_1}$ $(\theta_1 = \theta_3)$

The power in the transmitted wave $= \frac{E_t^2}{\eta_2}$

The component of this along the normal $= \frac{E_t^2 \cos\theta_2}{\eta_2}$

According to Law of Conservation of Energy

$$\frac{E_i^2 \cos\theta_1}{\eta_1} = \frac{E_r^2 \cos\theta_1}{\eta_1} + \frac{E_t^2 \cos\theta_2}{\eta_2}$$

Dividing With E_i^2

$$\frac{1}{\eta_1} \cos\theta_1 = \frac{E_r^2 \cos\theta_1}{E_i^2 \eta_1} + \frac{E_t^2 \cos\theta_2}{\eta_2 E_i^2}$$

$$\frac{E_r^2}{E_i^2} = 1 - \frac{\eta_1}{\eta_2} \frac{E_t^2}{E_i^2} \frac{\cos\theta_2}{\cos\theta_1}$$

$\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}$

$\eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}}$

If $\mu_1 = \mu_2 = \mu_0$

$$\frac{E_r^2}{E_i^2} = 1 - \frac{\sqrt{\varepsilon_2}\, E_t^2 \cos\theta_2}{\sqrt{\varepsilon_1}\, E_i^2 \cos\theta_1} \quad \text{— (1)}$$

Case 1 :- Horizontal (or) Perpendicular Polarization : the Electric vector 'E' is ⊥ar to the plane of incidence & Parallel to the boundary Surface.

According to the boundary conditions $E_i + E_r = E_t$

$$\frac{E_t}{E_i} = 1 + \frac{E_r}{E_i} \quad \text{— (2)}$$

Substitute eqn (2) in to eqn (1)

$$\frac{E_r^2}{E_i^2} = 1 - \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}} \left[1 + \frac{E_r}{E_i}\right]^2 \frac{\cos\theta_2}{\cos\theta_1}$$

$$1-\frac{E_r^2}{E_i^2} = \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\left[1+\frac{E_r}{E_i}\right]^2 \frac{\cos\theta_2}{\cos\theta_1}$$

$$\left[1+\cancel{\frac{E_r}{E_i}}\right]\left[1-\frac{E_r}{E_i}\right] = \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\left[1+\frac{E_r}{E_i}\right]^{\cancel{2}} \frac{\cos\theta_2}{\cos\theta_1}$$

$$1-\frac{E_r}{E_i} = \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\left[1+\frac{E_r}{E_i}\right]\frac{\cos\theta_2}{\cos\theta_1}$$

$$1-\frac{\sqrt{\varepsilon_2}\cos\theta_2}{\sqrt{\varepsilon_1}\cos\theta_1} = \frac{E_r}{E_i}\left[1+\frac{\sqrt{\varepsilon_2}\cos\theta_2}{\sqrt{\varepsilon_1}\cos\theta_1}\right]$$

$$\frac{E_r}{E_i} = \frac{\sqrt{\varepsilon_1}\cos\theta_1-\sqrt{\varepsilon_2}\cos\theta_2}{\sqrt{\varepsilon_1}\cos\theta_1+\sqrt{\varepsilon_2}\cos\theta_2}$$

from snell's Law

$$\frac{\sin\theta_1}{\sin\theta_2} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \qquad \sin\theta_2 = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\sin\theta_1$$

$$\cos\theta_2 = \sqrt{1-\frac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_1}$$

$$\leftarrow \frac{E_r}{E_i} = \frac{\sqrt{\varepsilon_1}\cos\theta_1-\sqrt{\varepsilon_2}\sqrt{1-\frac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_1}}{\sqrt{\varepsilon_1}\cos\theta_1+\sqrt{\varepsilon_2}\sqrt{1-\frac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_1}}$$

reflection coefficient

$$E_T = E_i + E_r$$

$$= \frac{\cos\theta_1-\sqrt{\frac{\varepsilon_2}{\varepsilon_1}-\sin^2\theta_1}}{\cos\theta_1+\sqrt{\frac{\varepsilon_2}{\varepsilon_1}-\sin^2\theta_1}} \quad —(2)$$

$$\frac{E_t}{E_i} = 1+\frac{E_r}{E_i}$$

↳ Transmission coefficient

Vertical Polarization :-

$$E_t = E_i + E_r$$

$$E_t\cos\theta_2 = E_i\cos\theta_1 - E_r\cos\theta_1$$

$$\frac{E_t}{E_i} = \frac{\cos\theta_1}{\cos\theta_2} - \frac{E_r}{E_i}\frac{\cos\theta_1}{\cos\theta_2} = \left[1-\frac{E_r}{E_i}\right]\frac{\cos\theta_1}{\cos\theta_2} \quad —(3)$$

Sub eqn(3) in eqn (1)

$$\frac{E_r^2}{E_i^2} = 1-\frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\left[1-\frac{E_r}{E_i}\right]^2\frac{\cos\theta_1^2}{\cos\theta_2^2}\times\frac{\cos\theta_2}{\cos\theta_1}$$

$$\left[1-\cancel{\frac{E_r}{E_i}}\right]\left[1+\frac{E_r}{E_i}\right] = \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\left[1-\frac{E_r}{E_i}\right]^{\cancel{2}}\frac{\cos\theta_1}{\cos\theta_2}$$

$$\frac{E_r}{E_i}\left[1+\frac{\sqrt{\varepsilon_2}\cos\theta_1}{\sqrt{\varepsilon_1}\cos\theta_2}\right] = \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\frac{\cos\theta_1}{\cos\theta_2}-1$$

$$\frac{E_r}{E_i} = \frac{\sqrt{\varepsilon_2}\cos\theta_1 - \sqrt{\varepsilon_1}\cos\theta_2}{\sqrt{\varepsilon_1}\cos\theta_2 + \sqrt{\varepsilon_2}\cos\theta_1}$$

Slide 3

From Snell's Law

$$\cos\theta_2 = \sqrt{1 - \frac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_1}$$

$$\frac{E_r}{E_i} = \frac{\sqrt{\varepsilon_2}\cos\theta_1 - \sqrt{\varepsilon_1}\sqrt{1-\varepsilon_1/\varepsilon_2 \sin^2\theta_1}}{\sqrt{\varepsilon_2}\cos\theta_1 + \sqrt{\varepsilon_1}\sqrt{1-\varepsilon_1/\varepsilon_2 \sin^2\theta_1}} \Rightarrow$$

divide by $\sqrt{\varepsilon_1}$

$$\frac{\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\cos\theta_1 - \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\sqrt{\frac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_1}}{\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\cos\theta_1 + \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\sqrt{\frac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_1}}$$

$$= \frac{\frac{\varepsilon_2}{\varepsilon_1}\cos\theta_1 - \sqrt{\frac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_1}}{\frac{\varepsilon_2}{\varepsilon_1}\cos\theta_1 + \sqrt{\frac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_1}}$$

$$\frac{E_t}{E_i} = 1 + \frac{E_r}{E_i}$$

Brewster angle:-

It is the angle of incidence for which there is no reflections. complete transmission takes place.

At the Brewster angle $E_r = 0$

For Vertical polarization:

$$\frac{E_r}{E_i} = \frac{\frac{\varepsilon_2}{\varepsilon_1}\cos\theta_1 - \sqrt{\frac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_1}}{\frac{\varepsilon_2}{\varepsilon_1}\cos\theta_1 + \sqrt{\frac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_1}} = 0$$

$$\left[\frac{\varepsilon_2}{\varepsilon_1}\right]^2\cos^2\theta_1 = \frac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_1$$

$$\left[\frac{\varepsilon_2}{\varepsilon_1}\right]^2\left[1 - \sin^2\theta_1\right] = \frac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_1$$

$$\sin^2\theta_1\left[1 - \frac{\varepsilon_2^2}{\varepsilon_1^2}\right] = \frac{\varepsilon_2}{\varepsilon_1} - \frac{\varepsilon_2^2}{\varepsilon_1^2}$$

$$\sin^2\theta_1 = \frac{\varepsilon_2\varepsilon_1^2 - \varepsilon_1\varepsilon_2^2}{\varepsilon_1\cancel{\varepsilon_1^2}} \times \frac{\cancel{\varepsilon_1^2}}{\varepsilon_1^2 - \varepsilon_2^2}$$

$$= \frac{\varepsilon_2\varepsilon_1 - \varepsilon_2^2}{\varepsilon_1^2 - \varepsilon_2^2} = \frac{\varepsilon_2(\cancel{\varepsilon_1 - \varepsilon_2})}{[\cancel{\varepsilon_1 - \varepsilon_2}][\varepsilon_1 + \varepsilon_2]}$$

$$\sin^2\theta_1 = \frac{\varepsilon_2}{\varepsilon_1 + \varepsilon_2}$$

$$\cos^2\theta_1 = \varepsilon_1 / \varepsilon_1 + \varepsilon_2$$

$$\tan\theta_1 = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \quad (\text{or}) \quad \theta_1 = \tan^{-1}\left[\frac{\varepsilon_2}{\varepsilon_1}\right]$$

For horizontal polarization:

$$\frac{E_r}{E_i} = \frac{\cos\theta_1 - \sqrt{\frac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_1}}{\cos\theta_1 + \sqrt{\frac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_1}}$$

5. APPLICATIONS

1. GSM based door control system can be used to eliminate the normal locks that are used at houses.
2. This system can be used at Bank lockers
3. This system can also be used for secure storage for medications, jewelry, weapons, documents, and other valuable or potentially harmful items.

capacitor and in many cases does not require a decoupling capacitor on the power supply. Such a practice should nevertheless be avoided, because noise produced by the timer or variation in power supply voltage might interfere with other parts of a circuit or influence its threshold voltages.

4.3.2 Pins:

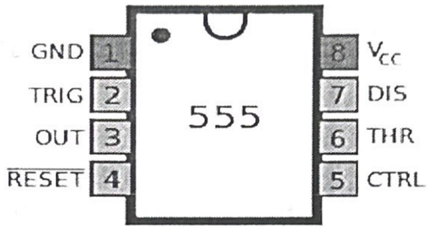


Fig no:4.3.2.1 :555 Pinout diagram

Pin Name		**Purpose**
1	GND	Ground reference voltage, low level (0 V)
2	TRIG	The OUT pin goes high and a timing interval starts when this input falls below 1/2 of CTRL voltage (which is typically 1/3 of V_{CC}, when CTRL is open).
3	OUT	This output is driven to approximately 1.7V below $+V_{CC}$ or GND.
4	RESET	A timing interval may be reset by driving this input to GND, but the timing does not begin again until RESET rises above approximately 0.7 volts. Overrides TRIG which overrides THR.
5	CTRL	Provides "control" access to the internal voltage divider (by default, 2/3 V_{CC}).
6	THR	The timing (OUT high) interval ends when the voltage at THR is greater than that at CTRL (2/3 V_{CC} if CTRL is open).
7	DIS	Open collector output which may discharge a capacitor between intervals. In phase with output.

Ex: consider a typical 16 guage insulated cable pair whose line parameters are $R = 42.1\ \Omega/km$, $G = 1.5\ \mu℧/km$, $C = 0.062\ \mu F/km$, $L = 1\ mH/km$

$R/L = 42.1 \times 10^3$; $\frac{G}{C} = 24.2$ $\quad R/L >> \frac{G}{C}$

Hence to make a line distortion less, either R/L is to be decreased (or) G/c is to be increased. There are three methods by which this can be acheived.

Method 1:- Reduction of the resistance 'R' results in reduction of R/L

disadvantage : reduction of 'R' is that it reduces Z_0 $\left[Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}} \right]$

This causes reflection of energy from the Load ends.

Advantage : I^2R loss in the line is reduced with decrease in 'R'.

$\vec{B} = \int_L + \nabla \times \frac{\mu_0 d\vec{l}}{4\pi R}$

Method 2: Increasing 'L' i.e the Inductance, the value of $\searrow R/L$ decreases. (Loading of Lines)

disadvantage : This requires additional amount of reactive power in the line.

Advantage : best Method because it can be easily acheived by changing the configuration of conductors.

$L \uparrow \quad \alpha \downarrow$ and hence reduces distortion

$\left[\alpha = \frac{R}{L} \sqrt{LC} \right]$

$L \uparrow \quad Z_0 \uparrow$ does not cause energy reflection from the Load.

$Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}}$

omsai

UNIT - 6 — GUIDED WAVES

Till now only uniform plane waves without any guiding surface have been considered in actual propagation. In actual causes propagation is by means of guided waves.

Guided waves : They are the waves that are guided along (or) over a conducting (or) dielectric surface.

Ex: waves along ordinary coaxial transmission lines, waves in waveguides.

Waves between parallel planes/plates :-

→ consider an EM wave propagating between a pair of parallel perfectly conducting planes of infinite extent along 'y' and 'z' directions. In order to determine the EM field configuration between the planes Maxwell's equations will be solved.

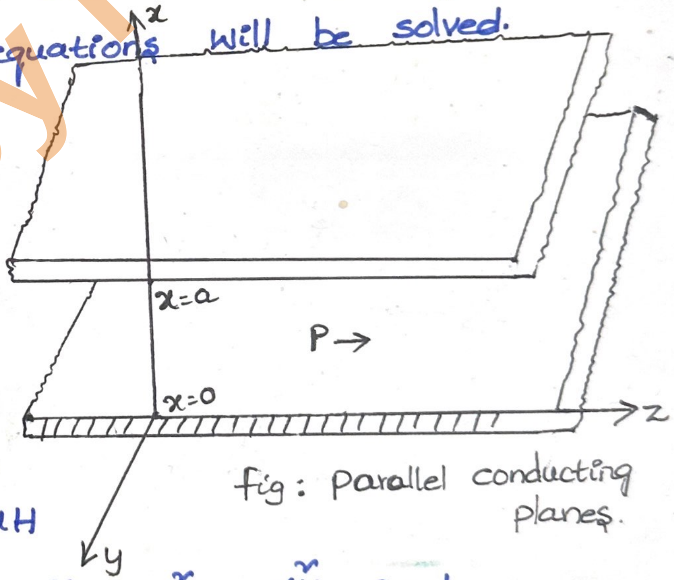


fig : Parallel conducting planes.

consider Maxwell's first eqn

$$\nabla \times H = J + \dot{D}$$

$$= \sigma E + \varepsilon \dot{E}$$

$$= \sigma E + j\omega \varepsilon E$$

$$= E(\sigma + j\omega\varepsilon)$$

$$\nabla \times E = -\dot{B} = -\mu \dot{H} = -j\omega\mu H$$

The wave equations are $\nabla^2 E = \gamma^2 E$, $\nabla^2 H = \gamma^2 H$ and

$$\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon)$$

In rectangular coordinates the above equations for non-conducting region (dielectric) becomes

In dielectric medium (perfect) $\boxed{\sigma = 0}$

$$\nabla \times H = \begin{vmatrix} \vec{x} & \vec{y} & \vec{z} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ H_x & H_y & H_z \end{vmatrix}$$

$$\vec{x}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) - \vec{y}\left(\frac{\partial H_z}{\partial x} - \frac{\partial H_x}{\partial z}\right) + \vec{z}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)$$

$$= j\omega\varepsilon E = j\omega\varepsilon\left(\vec{x}E_x + \vec{y}E_y + \vec{z}E_z\right) \qquad \text{when } \sigma = 0$$

Equating x, y and z components

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = j\omega\varepsilon E_x \longrightarrow (1)$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y \longrightarrow (2)$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\varepsilon E_z \longrightarrow (3)$$

III'y $\nabla \times E = -j\omega\mu H$

$$\frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} = j\omega\mu H_x \longrightarrow (4)$$

$$\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} = j\omega\mu H_y \longrightarrow (5)$$

$$\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} = j\omega\mu H_z \longrightarrow (6)$$

Expanding the wave eqn's $\quad \gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon)$

$$\gamma^2 = j\omega\mu \times j\omega\varepsilon = -\omega^2\mu\varepsilon$$

The wave eqn $\nabla^2 E = \gamma^2 E$

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} = -\omega^2\mu\varepsilon E \longrightarrow (7)$$

III'y

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} + \frac{\partial^2 H}{\partial z^2} = -\omega^2\mu\varepsilon H \longrightarrow (8)$$

$\Rightarrow \gamma = \alpha + j\beta$

It is assumed that the Propagation of EM wave is along z-direction. The variation along the direction of propagation is expressed as $e^{-\gamma z} \cdot e^{j\omega t} = e^{-(\alpha+j\beta)z} \cdot e^{j\omega t}$

$$= e^{-\alpha z} \cdot e^{j(\omega t - \beta z)}$$

→ If $\alpha = 0$, there is no attenuation of the wave as it propagates.

→ If $\beta = 0$, there is no wave motion but only an exponential decrease in amplitude.

Since the space between the planes is infinite in extent in the y-direction, there are no boundary conditions to be met in the direction, and it can be assumed that the field is uniform (or) constant in the y-direction. $\therefore \frac{\partial}{\partial y} = 0$

Let $H_y = H_y^0 e^{-\gamma z}$ The direction of propagation along the z-direction

$$\frac{\partial}{\partial z}(H_y) = \frac{\partial}{\partial z} H_y^0 e^{-\gamma z}$$

$$= -\gamma H_y^0 e^{-\gamma z} = -\gamma H_y$$

$$\frac{\partial}{\partial z}(\text{any component}) = -\gamma \times (\text{that component})$$

The Maxwell's equations become $\frac{\partial}{\partial y} = 0$

from eqn ① can be written as

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = j\omega\varepsilon E_x \Rightarrow 0 - (-\gamma H_y) = j\omega\varepsilon E_x$$

$$\gamma H_y = j\omega\varepsilon E_x \quad \text{— (9)}$$

2 → $$-\gamma H_x - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y \quad \text{— (10)}$$

3 → $$\frac{\partial H_y}{\partial x} = j\omega\varepsilon E_z \quad \text{— (11)}$$

← 4 $$-\gamma E_y = j\omega\mu H_x \quad \text{— (12)}$$

← 5 $$\frac{\partial E_z}{\partial x} + \gamma E_x = j\omega\mu H_y \quad \text{— (13)}$$

← 6 $$-\frac{\partial E_y}{\partial x} = j\omega\mu H_z \quad \text{— (14)}$$

7 → $$\frac{\partial^2 E}{\partial x^2} + 0 + \gamma^2 E = -\omega^2\mu\varepsilon E$$

$$\Rightarrow \frac{\partial^2 E}{\partial x^2} + \gamma^2 E = -\omega^2\mu\varepsilon E \quad \text{— (15)}$$

III^ly $$\frac{\partial^2 H}{\partial x^2} + \gamma^2 H = -\omega^2\mu\varepsilon H \quad \text{— (16)}$$

from eqn (9) $\gamma H_y = j\omega\varepsilon E_x$

$$E_x = \left(\frac{\gamma}{j\omega\varepsilon}\right) H_y \quad -(18)$$

from eqn (13)

$$j\omega\mu H_y = \frac{\partial E_z}{\partial x} + \gamma E_x \quad -(17)$$

Substitute eqn (18) (or) E_x in eqn (17) we get

$$j\omega\mu H_y = \frac{\partial E_z}{\partial x} + \gamma\left(\frac{\gamma}{j\omega\varepsilon}\right) H_y$$

$$= \frac{\partial E_z}{\partial x} + \frac{\gamma^2}{j\omega\varepsilon} H_y$$

$$\left(j\omega\mu - \frac{\gamma^2}{j\omega\varepsilon}\right) H_y = \frac{\partial E_z}{\partial x}$$

$$\left(\frac{-\omega^2\mu\varepsilon - \gamma^2}{j\omega\varepsilon}\right) H_y = \frac{\partial E_z}{\partial x}$$

Let $h^2 = \gamma^2 + \omega^2\mu\varepsilon$

$$^{***}\underline{H_y} = \frac{\partial E_z}{\partial x} \times \frac{-j\omega\varepsilon}{h^2} \quad -(19)$$

Substitute in eqn (18)

$$^{***}\underline{E_x} = \frac{\gamma}{\cancel{j\omega\varepsilon}} \times \frac{-\cancel{j\omega\varepsilon}}{h^2} \frac{\partial E_z}{\partial x} = \frac{-\gamma}{h^2}\frac{\partial E_z}{\partial x} \quad -(20)$$

from eqn (10)

$$-\gamma H_x - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y$$

$$\frac{\gamma^2}{j\omega\mu} \overset{E_y}{H_x} - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y$$

$$E_y\left[\frac{\gamma^2}{j\omega\mu} - j\omega\varepsilon\right] = \frac{\partial H_z}{\partial x}$$

$$E_y\left[\frac{\gamma^2 + \omega^2\mu\varepsilon}{j\omega\mu}\right] = \frac{\partial H_z}{\partial x}$$

$$^{***}\underline{E_y} = \frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial x} \quad -(21)$$

from eqn (12)

$$H_x = \frac{-\gamma}{j\omega\mu} E_y$$

$$^{***}\underline{H_x} = \frac{-\gamma}{j\omega\mu} \times \frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial x}$$

$$= \frac{-\gamma}{h^2}\frac{\partial H_z}{\partial x} \quad -(22)$$

In general both E_z and H_z components would be present. It is convenient to divide the solution in to two sets

(i) $H_z = 0$ and $E_z \neq 0$

$\downarrow$ direction of propagation along the z-direction

The remaining components of the magnetic field (H) is tranverse to the direction of wave propagation. Such waves are called transverse magnetic wave (TM) waves.

(ii) $E_z = 0$ and $H_z \neq 0$

such waves are called tranverse electric waves (TE waves)

Transverse Electric Waves ($E_z = 0$) :-

When $E_z = 0 \Rightarrow E_x = 0,\ H_y = 0$

$\overline{E_y \neq 0}$, because $H_z \neq 0$

from eqn (15)

$$\frac{\partial^2 E}{\partial x^2} + \gamma^2 E = -\omega^2 \mu \varepsilon E$$

$$\frac{\partial^2 E}{\partial x^2} = -\gamma^2 E - \omega^2 \mu \varepsilon E$$

$$= -E(\gamma^2 + \omega^2 \mu \varepsilon) = -h^2 E$$

$$\frac{\partial^2 E}{\partial x^2} = -h^2 E$$

considering only the y-component since only E_y is existing

$$\Rightarrow \frac{\partial^2 E_y}{\partial x^2} = -h^2 E_y$$

but $E_y = E_y^0(x)\, e^{-\gamma z}$

$$\frac{\partial^2}{\partial x^2}\left(E_y^0(x)\cancel{e^{-\gamma z}}\right) = -h^2\left(E_y^0(x)\cancel{e^{-\gamma z}}\right)$$

$$\frac{\partial^2}{\partial x^2} E_y^0(x) = -h^2 E_y^0(x)$$

This is a 2^{nd} order differential equation of simple Harmonic motion. Its solution can be written as

$$E_y^0(x) = c_1 \sin hx + c_2 \cos hx$$

where c_1 & c_2 are arbitrary complex constants.

$$E_y = E_y^0(x)\, e^{-\gamma z}$$

$$= (c_1 \sin hx + c_2 \cos hx)\, e^{-\gamma z}$$

For the parallel-plane wave guide of fig: the boundary conditions are quite simple. They require that the tangential component of E be zero at the surface of the perfect conductors for all values of z and time. This requires that

$$\left.\begin{aligned} E_y = 0 \quad \text{at} \quad x = 0 \\ E_y = 0 \quad \text{at} \quad x = a \end{aligned}\right\} \text{for all values of } z \quad \text{(boundary conditions)}$$

'a' is the distance seperation between the plates.

→ $E_y = 0$ at $x = 0$

$$0 = (c_1 (0) + c_2 \cos(0))\, e^{-\gamma z} \Rightarrow c_2 e^{-\gamma z} = 0$$

$$\Rightarrow \boxed{c_2 = 0}$$

$$\therefore E_y = c_1 \sin hx\, e^{-\gamma z}$$

→ $E_y = 0$ at $x = a$

$$0 = c_1 \sin ha\, e^{-\gamma z} \Rightarrow c_1 \sin ha = 0$$

$$\sin ha = 0$$

$$ha = \sin^{-1}(0)$$

$$ha = m\pi$$

$$\boxed{h = \frac{m\pi}{a}}$$

Where $m = 1, 2, 3, \dots$

$$\therefore E_y = c_1 \sin\left(\frac{m\pi}{a}\right)x\, e^{-\gamma z} \quad \text{— (23)}$$

from eqn (12)

$$j\omega\mu H_x = -\gamma E_y$$

$$H_x = \frac{-\gamma}{j\omega\mu} E_y$$

$$= \frac{-\gamma}{j\omega\mu} c_1 \sin\left(\frac{m\pi}{a}\right)x\, e^{-\gamma z} \quad \text{— (24)}$$

from eqn (14)

$$j\omega\mu H_z = \frac{-d}{dx} E_y$$

$$H_z = \frac{1}{j\omega\mu} \frac{-d}{dx} c_1 \sin\left(\frac{m\pi}{a}\right)x\, e^{-\gamma z}$$

$$= \frac{1 \times (-c_1)}{j\omega\mu} \times \cos\left(\frac{m\pi}{a}\right)x\, e^{-\gamma z} \times \frac{m\pi}{a}$$

$$= \frac{-m\pi}{j\omega\mu a} c_1 \cos\left(\frac{m\pi}{a}\right)x\, e^{-\gamma z} \quad \text{— (25)}$$

Guided Waves

Consider Maxwell's first eqn

$$\nabla \times H = J + \dot{D} = \sigma E + \epsilon \dot{E} = \sigma E + j\omega\epsilon E = E(\sigma + j\omega\epsilon) \quad — (1)$$

$$\nabla \times E = -\dot{B} = -\mu \dot{H} = -j\omega\mu H \quad — (2)$$

The wave equations are $\nabla^2 E = \gamma^2 E$, $\nabla^2 H = \gamma^2 H$ and

$$\gamma^2 = j\omega\mu(\sigma + j\omega\epsilon)$$

from eqn (1)

for non conducting medium (or) dielectric medium $\boxed{\sigma = 0}$

$$\nabla \times H = \begin{vmatrix} \vec{x} & \vec{y} & \vec{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = j\omega\epsilon E \quad (\sigma = 0)$$

$$\vec{x}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) - \vec{y}\left(\frac{\partial H_z}{\partial x} - \frac{\partial H_x}{\partial z}\right) + \vec{z}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) = j\omega\epsilon(E_x\vec{x} + E_y\vec{y} + E_z\vec{z})$$

Equating x, y and z components

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = j\omega\epsilon E_x \quad — (3)$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j\omega\epsilon E_y \quad — (4)$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\epsilon E_z \quad — (5)$$

III'ly $\nabla \times E = -j\omega\mu H$

$$\frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} = j\omega\mu H_x \quad — (6)$$

$$\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} = j\omega\mu H_y \quad — (7)$$

$$\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} = j\omega\mu H_z \quad — (8)$$

Expanding the wave eqn's

$$\gamma^2 = j\omega\mu(\sigma + j\omega\epsilon) = j\omega\mu(j\omega\epsilon) = -\omega^2\mu\epsilon$$

The wave eqn $\nabla^2 E = \gamma^2 E$

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} = -\omega^2\mu\epsilon E \quad — (9)$$

III'ly

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} + \frac{\partial^2 H}{\partial z^2} = -\omega^2\mu\epsilon H \quad — (10)$$

$\gamma = \alpha + j\beta$

Let $H_y = H_y^0 e^{-\gamma z}$ direction of propagation along the z- direction.

$$\frac{\partial}{\partial z}(H_y) = \frac{\partial}{\partial z}\left[H_y^0 e^{-\gamma z}\right] \qquad \frac{\partial}{\partial y} = 0$$

$$= -\gamma H_y^0 e^{-\gamma z} = -\gamma H_y$$

The Maxwell's eqn's become

eqn ③ $\rightarrow$ $0 - (-\gamma H_y) = j\omega\varepsilon E_x \rightarrow \gamma H_y = j\omega\varepsilon E_x$ — ⑪

eqn ④ $\rightarrow$ $-\gamma H_x - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y$ — ⑫

eqn ⑤ $\rightarrow$ $\frac{\partial H_y}{\partial x} = j\omega\varepsilon E_z$ — ⑬

eqn ⑥ $\rightarrow$ $-\gamma E_y = j\omega\mu H_x$ — ⑭

eqn ⑦ $\rightarrow$ $\frac{\partial E_z}{\partial x} + \gamma E_x = j\omega\mu H_y$ — ⑮

eqn ⑧ $\rightarrow$ $-\frac{\partial E_y}{\partial x} = j\omega\mu H_z$ — ⑯

eqn ⑨ $\rightarrow$ $\frac{\partial^2 E}{\partial x^2} + \gamma^2 E = -\omega^2\mu\varepsilon E$ — ⑰

eqn ⑩ $\rightarrow$ $\frac{\partial^2 H}{\partial x^2} + \gamma^2 H = -\omega^2\mu\varepsilon H$ — ⑱

from eqn ⑪

*** $\boxed{E_x = \frac{\gamma H_y}{j\omega\varepsilon}}$ — ⑲

Substitute eqn ⑲ in eqn ⑮ we get

$$\frac{\partial E_z}{\partial x} + \frac{\gamma^2 H_y}{j\omega\varepsilon} = j\omega\mu H_y$$

$$\frac{\partial E_z}{\partial x} = j\omega\mu H_y - \frac{\gamma^2 H_y}{j\omega\varepsilon} = H_y\left[\frac{-\omega^2\mu\varepsilon - \gamma^2}{j\omega\varepsilon}\right]$$

Let $h^2 = \gamma^2 + \omega^2\mu\varepsilon$

$$H_y = \frac{-j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial x} \quad \text{— (20)}$$

Substitute in eqn (19)

$$E_x = \frac{-\gamma}{h^2}\frac{\partial E_z}{\partial x} \quad \text{— (21)}$$

From eqn (14)

$$H_x = \frac{-\gamma}{j\omega\mu} E_y \quad \text{— (22)}$$

Substitute eqn (22) in eqn (12)

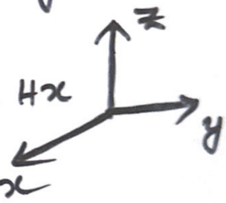
$$\frac{-\gamma^2}{j\omega\mu}E_y - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y$$

$$\Rightarrow E_y = \frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial x} \quad \text{— (23)}$$

Transverse Electric Wave (TE Wave)

The Electric field is purely tranverse to the direction of propagation but the magnetic field is not purely tranverse

i.e $E_z = 0$ $H_z \neq 0$



from eqn (17) ↳ $E_x = 0$, $H_y = 0$ only E_y & H_x are present

$$\frac{\partial^2 E}{\partial x^2} + \gamma^2 E = -\omega^2\mu\varepsilon E$$

$$\frac{\partial^2 E}{\partial x^2} = -\gamma^2 E - \omega^2\mu\varepsilon E$$

$$= -E(\gamma^2 + \omega^2\mu\varepsilon) = -h^2 E$$

$$\frac{\partial^2 E}{\partial x^2} = -h^2 E$$ considering only the y-component

$$\frac{\partial^2 E_y}{\partial x^2} = -h^2 E_y \Rightarrow \frac{d^2}{dx^2}\left[E_y^0(x)\, e^{-\gamma z}\right] = -h^2\left[E_y^0(x)\, e^{-\gamma z}\right]$$

$$\Rightarrow \frac{d^2}{dx^2} E_y^0(x) = -h^2 E_y^0(x)$$

$D^2 + h^2 = 0$

$D^2 = -h^2$

$D = \pm jh$

$$E_y^0(x) = C_1 \sin hx + C_2 \cos hx$$

Where C_1 & C_2 are arbitrary complex constants

$$\therefore E_y = E_y^0(x)\, e^{-\gamma z}$$

$$= \left[C_1 \sin hx + C_2 \cos hx\right] e^{-\gamma z} \quad \text{— (24)}$$

$\left.\begin{array}{ll} E_y = 0 & \text{at } x = 0 \\ E_y = 0 & \text{at } x = a \end{array}\right\}$ for all values of z

[boundary conditions]

substitute $E_y = 0$ at $x = 0$ in eqn (24)

$$0 = C_2 e^{-\gamma z} \Rightarrow \boxed{C_2 = 0}$$

$$\therefore E_y = C_1 \sinh x\, e^{-\gamma z}$$

$E_y = 0$ at $x = a$

$$0 = C_1 \sin ha\, e^{-\gamma z}$$

$$\sin ha = 0$$

$$ha = \sin^{-1}(0) = m\pi \Rightarrow \boxed{h = \frac{m\pi}{a}}$$

Where $m = 1, 2, 3, \ldots$

$$\therefore E_y = C_1 \sin\left[\frac{m\pi}{a}\right] x\, e^{-\gamma z} \quad \text{— (25)}$$

from eqn (16)

$$\frac{-\partial E_y}{\partial x} = j\omega\mu H_z$$

$$H_z = \frac{-1\, C_1}{j\omega\mu} \cos\left[\frac{m\pi}{a}\right] x\, e^{-\gamma z} \left[\frac{m\pi}{a}\right]$$

$$= \frac{-m\pi}{j\omega\mu a} C_1 \cos\left[\frac{m\pi}{a}\right] x\, e^{-\gamma z} \quad \text{— (26)}$$

III'ly $$H_x = \frac{-\gamma}{j\omega\mu} C_1 \sin\left[\frac{m\pi}{a}\right] x\, e^{-\gamma z} \quad \text{— (27)}$$

Tranverse magnetic waves :

The magnetic field is tranverse to the direction of propagation but the electric field is not purely tranverse i.e $H_z = 0 \Rightarrow H_x = E_y = 0$ only E_z, E_x & H_y will be present

$$\frac{\partial^2 H}{\partial x^2} = -h^2 H$$

$$\frac{\partial^2 H_y}{\partial x^2} = -h^2 H_y \Rightarrow H_y = \left[C_3 \sinh x + C_4 \cosh x\right] e^{-\gamma z}$$

From eqn (20)

$$H_y = -\frac{\partial E_z}{\partial x} \times \frac{-j\omega\varepsilon}{h^2} \Rightarrow \frac{\partial E_z}{\partial x} = H_y \times \frac{-h^2}{j\omega\varepsilon}$$

$$\frac{\partial E_z}{\partial x} = \left[C_3 \sinh x + C_4 \cosh x\right] e^{-\gamma z} \times \frac{-h^2}{j\omega\varepsilon}$$

Integrate on both sides w.r.to x

$$E_z = \frac{-h^2 e^{-rz}}{j\omega\varepsilon}\left[\frac{-c_3\cosh x + c_4 \sinh x}{h}\right]$$

$$= \frac{h\, e^{-rz}}{j\omega\varepsilon}\left[c_3\cosh x - c_4\sinh x\right]$$

$E_z = 0$ at $x = 0$ [1st boundary condition]

$$0 = \frac{h e^{-rz}}{j\omega\varepsilon}[c_3] \Rightarrow \boxed{c_3 = 0}$$

$$\therefore E_z = \frac{h e^{-rz}}{j\omega\varepsilon}\left[-c_4\sin hx\right]$$

$E_z = 0$ at $x = a$ [2nd B.C]

$$0 = \frac{h e^{-rz}}{j\omega\varepsilon}\left[-c_4 \sin ha\right] \Rightarrow \sin ha = 0$$

$$h = \left[\frac{m\pi}{a}\right]$$

$$\therefore E_z = \frac{m\pi e^{-rz}}{j\omega\varepsilon a}\left[-c_4\sin\left(\frac{m\pi}{a}\right)x\right]$$

Substitute E_z in H_y eqn we get

$$H_y = c_4 e^{-rz}\cos\left(\frac{m\pi}{a}\right)x$$

$$\& \; E_x = \frac{c_4 r}{j\omega\varepsilon} e^{-rz}\cos\left(\frac{m\pi}{a}\right)x$$